Chapter 6: Stochastic (Random) Processes

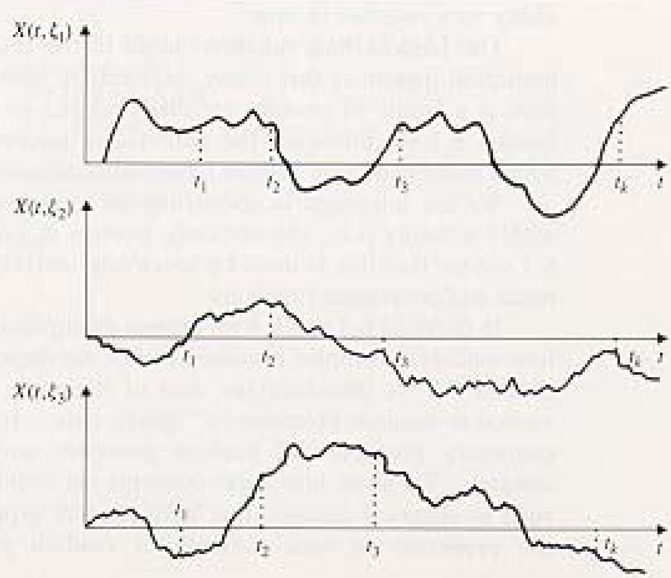
Let outcomes ξ from S be such that, for $\xi \in$ S we assign a function of time according to some rule:

$$X(t,\xi) \quad \text{where } t \in I$$

1) The graph of $X(t,\xi)$ for a fixed ξ is called a realization.
2) For each fixed t_k the set $X(t_k,\xi)$ is a r.v.
   $\Rightarrow$ Indexed family of r.v. $\Rightarrow$ Stochastic Process

- If the index set "I" is If "I" is continuous, then it is a continuous-time stochastic process.
- If discrete-time then, we have a discrete-time stochastic process.

FIGURE 6.1
Several realizations of a random process



Ex: 6.1 Random Binary Sequence

ξ selected randomly in interval S = [0,1] $b_1 b_2 \ldots$ binary expansion of ξ , then

$$\xi = \sum_{i=1}^{\infty} b_i 2^{-i} \; ; \quad \text{where } b_i \in \{0,1\}$$

Define $X(t,\xi) = b_n$ n = 1,2,… The result is a sequence of binary numbers.

Ex: 6.3 Find P[X(1,ξ)=0] and P[X(1,ξ)=0 and X(2,ξ)=1]

$$P[X(1,\xi)=0] = P[0 \le \xi < 1/2] = 1/2$$

$$P[X(1,\xi)=0 \text{ and } X(2,\xi)=1] = P[1/4 \le \xi < 1/2] = 1/4$$

Sequence of k bits has subinterval of length 2^{-k} .

Ex: 6.2 Random Sinusoids

FIGURE 6.2a
Sinusoid with random amplitude.

ξ in interval S = [-1,1]
Define $X(t,\xi) = \xi\cos(2\pi t)$
$-\infty < t < \infty$
Amplitude versions

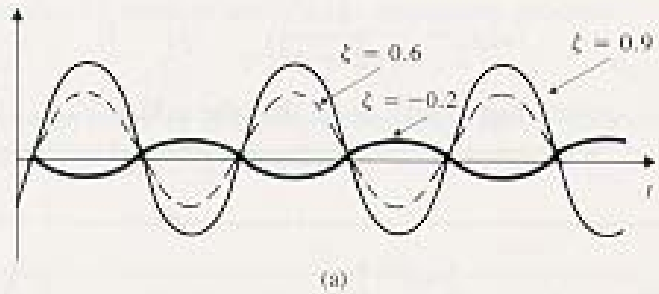
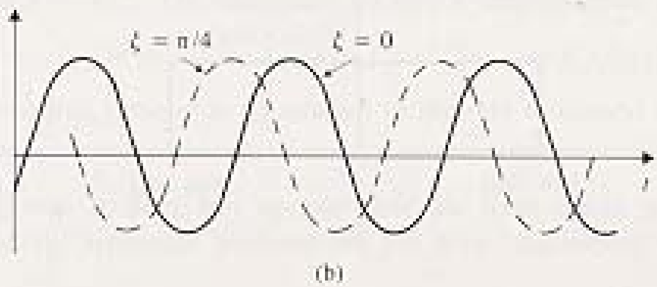


FIGURE 6.2b
Sinusoid with random phase.

ξ in interval S = [-π,π]
Define $Y(t,\xi) = \text{Cos}(2\pi t+\xi)$

Time-shifted versions



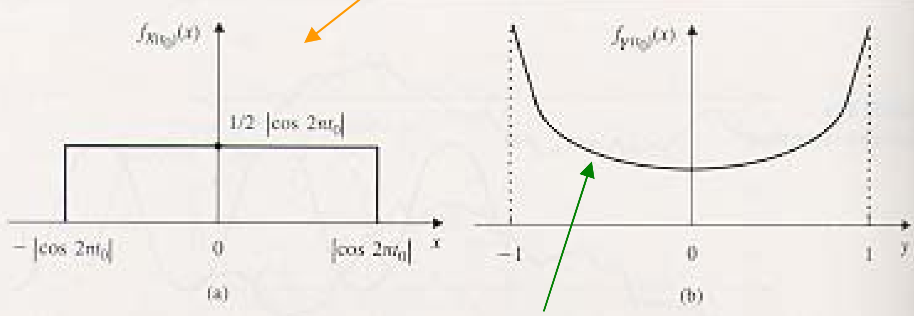
Ex: 6.4 Find pdf of $X(t_0,\xi)$ and $Y(t_0,\xi)$ of Ex: 6.2

- If $\cos(2\pi t_0) = 0$, $X(t_0,\xi) = 0 \quad \Rightarrow \quad f_{X(t_0)}(x) = \delta(x)$
- Else, $X(t_0,\xi)$ is uniformly distributed in $[-\cos(2\pi t_0), \cos(2\pi t_0)]$, since $X(t_0,\xi)$ is uniformly distributed in [-1,1]

$$f_{X(t_0)}(x) = \begin{cases} 0 & o.w. \\ 1/2|\cos(2\pi t_0)| & x \le |\cos(2\pi t_0)| \end{cases}$$

Note: pdf of $X(t_0,\xi)$ depends on t_0.

FIGURE 6.3
(a) pdf of sinusoid with random amplitude, and (b) pdf of sinusoid with random phase.



$Y(t_0,\xi)$ has an arcsine distribution (see Ex: 3.28).

$$f_{Y(t_0)}(y) = \frac{1}{\pi\sqrt{1-y^2}} \qquad |y| < 1$$

Note: pdf of $Y(t_0,\xi)$ does not depend on t_0.

Random Process Specification:

Let $X_1, X_2, \ldots, X_k$ be k r.v.'s obtained by sampling a Random Process $X(t,\xi)$ at times $t_1, \ldots, t_k$

$$X_1 = X(t_1,\xi)\ ,\ \ X_2 = X(t_2,\xi)\ ,\ldots,\ X_k = X(t_k,\xi)$$

Then a stochastic (random) process is specified by the collection of k^{th} order joint cdf:

$$F_{X_1 \ldots X_k}(x_1, x_2, \ldots x_k) = P\left[X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k\right]$$

If Stochastic Process is discrete then pmf can be used to specify Stochastic Process

$$p_{X_1 \ldots X_k}(x_1, x_2, \ldots x_k) = P\left[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\right]$$

If Stochastic Process is continuous-valued the pdf can be used to specify Stochastic Process:

$$f_{X_1 \ldots X_k}(x_1, x_2, \ldots x_k)$$

A Stochastic Process X(t) has independent increments if for any k and any choice of sampling instants:

$t_1 < t_2 < \ldots < t_k$, the random variables
$X(t_2) - X(t_1) \ldots X(t_k) - X(t_{k-1})$ are independent

Then the joint pdf (pmf) of $X(t_1) \ldots X(t_k)$ is given by the product of marginal pdf (pmf).

A Stochastic Process is **Markov** if the future of the process given the present is independent of the past:

$$f_{X(t_k)}\left(x_k \mid X(t_{k-1}) = x_{k-1}, \ldots, X(t_1) = x_1\right)$$
$$= f_{X(t_k)}\left(x_k \mid X(t_{k-1}) = x_{k-1}\right)$$

If X(t) is continuous, but for discrete X(t) the expression becomes

$$P\left[X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}, \ldots, X(t_1) = x_1\right]$$
$$= P\left[X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}\right]$$

Mean function:

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)dx$$

In general the mean function is a function of time.

Autocorrelation function (joint moment):

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X(t_1)X(t_2)}(x,y)dxdy$$

Autocovariance function:

$$C_X(t_1,t_2) = E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))]$$
$$= R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

Variance of X(t):

$$\sigma^2{}_{X(t)} = VAR[X(t)] = E\left[(X(t_1) - m_X(t_1))^2\right] = C_X(t,t)$$

Correlation Coefficient:

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)C_X(t_2,t_2)}}$$ with the property: $|\rho_X(t_1,t_2)| \le 1$

Ex: 6.6 Let X(t) = Acos2πt. Find mean, autocorrelation and autocovariance

$$m_X(t) = E[A\cos 2\pi t] = E[A]\cos 2\pi t$$

Note: The mean function is time-dependent.

$$R_X(t_1,t_2) = E[A\cos 2\pi t_1 A\cos 2\pi t_2] = E\left[A^2\right]\cos 2\pi t_1 \cos 2\pi t_2$$

$$C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2) = \left\{E\left[A^2\right] - E[A]^2\right\}\cos 2\pi t_1 \cos 2\pi t_2$$
$$= VAR[A]\cos 2\pi t_1 \cos 2\pi t_2$$

Ex: 6.7 Let X(t) = cos(wt+θ) , where θ is uniformly distributed in (-π,π). Let us find mean, autocorrelation and autocovariance.

$$m_X(t) = E[\cos(wt+\theta)] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(wt+\theta)d\theta = 0$$

$$C_X(t_1,t_2) = R_X(t_1,t_2) = E[\cos(wt_1+\theta)\cos(wt_2+\theta)]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\{\cos(w(t_1-t_2)) + \cos(w(t_1+t_2)+2\theta)\}d\theta$$

$$= \frac{1}{2}\cos w(t_1-t_2) \qquad \textit{See Appendix A}$$

Note: $m_X(t)$ is constant and $C_X(t_1,t_2)$ depends only on $|t_1-t_2|$.

Gaussian Random Process: X(t) is a Gaussian S.P. if the samples $X_1 = X(t_1)$, …, $X_k = X(t_k)$ are jointly Gaussian with

$$f_{X_1X_2\ldots X_k}(x_1,\ldots,x_k) = \frac{1}{(2\pi)^{k/2}|K|^{1/2}}\exp\left\{-\frac{1}{2}(\underline{x}-\underline{m})^T K^{-1}(\underline{x}-\underline{m})\right\}$$

where

$$\underline{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ m_X(t_k) \end{bmatrix} \qquad K = \begin{bmatrix} C_X(t_1,t_1) & C_X(t_1,t_2) & \cdots & C_X(t_1,t_k) \\ C_X(t_2,t_1) & C_X(t_2,t_2) & \cdots & C_X(t_2,t_k) \\ \vdots & \vdots & & \vdots \\ C_X(t_k,t_1) & \cdots & & C_X(t_k,t_k) \end{bmatrix}$$

Ex 6.8 X_n is iid Gaussian r.v. with m and σ^2, then

$$K = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^2 \end{bmatrix} = \sigma^2 I \qquad \text{Because: } C_X(t_i,t_j) = \sigma^2\delta_{ij}$$

Then:

$$f_{X_1X_2\ldots X_k}(x_1,\ldots,x_k) = \frac{1}{(2\pi\sigma^2)^{k/2}}\exp\left\{-\sum_{i=1}^{k}(\underline{x_i}-\underline{m})^2/2\sigma^2\right\}$$

$$= f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_k}(x_k)$$

Two or more variable Random Process:

1. For a pair of S.P. X(t) and Y(t′) all possible joint density functions must be specified for all choices of $t_1,\ldots,t_k$ and $t_1',\ldots,t_k'$.
2. X(t) and Y(t′) are independent iff the vector r.v. **X** and **Y** are **independent** for all k, j and all choices of $t_1,\ldots,t_k$, $t_1',\ldots,t_k'$.
3. **Crosscorrelation:** $R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)]$
   X(t) and Y(t) processes are orthogonal if $R_{XY}(t_1,t_2) = 0$ for all t_1 and t_2
4. **Cross-Covariance:** $C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - m_X(t_1)m_Y(t_2)$
   X(t) and Y(t) are uncorrelated if $C_{XY}(t_1,t_2) = 0$ for all t_1 and t_2

Ex: 6.9 Given a process with $X(t) = \cos(wt+\theta)$ *and* $Y(t) = \sin(wt+\theta)$, where θ is uniformly distributed in $[-\pi,\pi]$. Find cross-covariance.

$$R_{XY}(t_1,t_2) = E[\cos(wt_1+\theta)\sin(wt_2+\theta)]$$
$$= E\left[-\frac{1}{2}\sin(w(t_1-t_2)) + \frac{1}{2}\sin(w(t_1+t_2)+2\theta)\right]$$
$$= -\frac{1}{2}\sin(w(t_1-t_2))$$

Ex: 6.10 Given an additive noise channel with a model: $Y(t) = X(t) + N(t)$ Find cross-correlation. Assume that X(t) and N(t) are independent

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = E[X(t_1)\{X(t_2)+N(t_2)\}]$$
$$R_{XY}(t_1,t_2) = E[X(t_1)X(t_2)] + E[X(t_1)N(t_2)]$$
$$= R_X(t_1,t_2) + E[X(t_1)]E[N(t_2)]$$
$$= R_X(t_1,t_2) + m_X(t_1)m_N(t_2)$$

Independent

Examples of Discrete-Time Stochastic Processes:

Given iid Stochastic Process: X_n : discrete iid r.v. with common, m, σ^2
Then, X_n – sequence is called iid R.P. and for any time instants $n_1, \ldots, n_k$

$$F_{X_1 \ldots X_k}(x_1,\ldots,x_k) = P[X_1 \le x_1,\ldots,X_k \le x_k]$$
$$= F_X(x_1)F_X(x_2)\cdots F_X(x_k)$$

The mean of iid S.P.:

$m_X(n) = E[X_n] = m$ for all n; Constant mean

$$\text{if} \quad n_1 \neq n_2: \quad C_X(n_1, n_2) = E\left[(X_{n_1} - m)(X_{n_2} - m)\right]$$
$$= E\left[X_{n_1} - m\right]E\left[X_{n_2} - m\right] = 0$$
$$\text{if} \quad n_1 = n_2: \quad C_X(n, n) = E\left[(X_n - m)^2\right] = \sigma^2$$

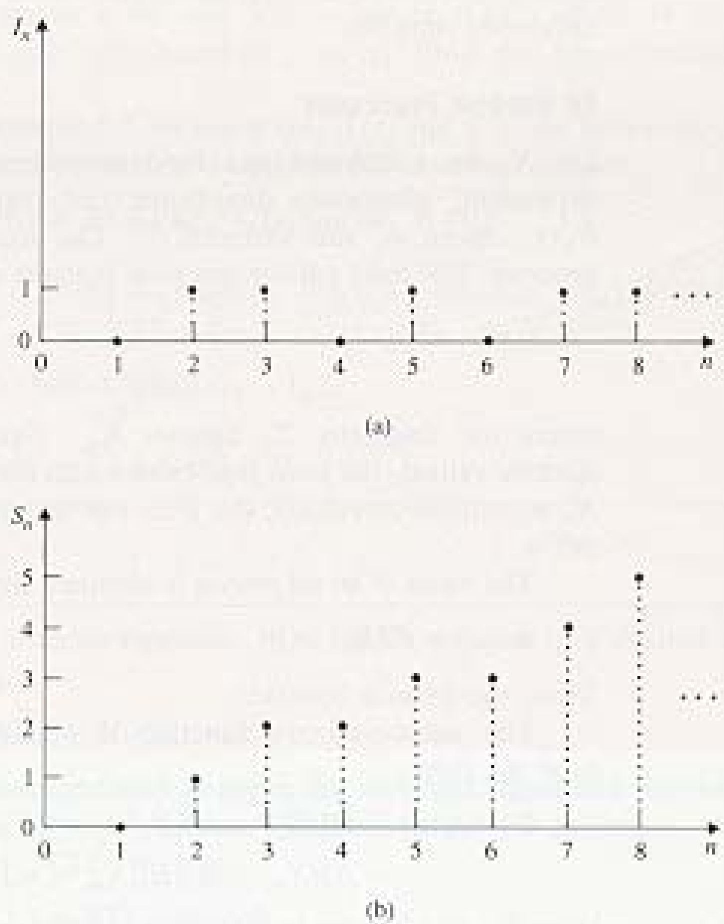
Because: $C_X(n_1, n_2) = R_X(n_1, n_2) - m^2$, which results in:

$$C_X(n_1, n_2) = R_X(n_1, n_2) - m^2$$
$$\Rightarrow R_X(n_1, n_2) = C_X(n_1, n_2) + m^2$$

Ex: 6.11 Bernoulli R.P. : i.i.d. Bernoulli R.V. I_n from a set {0,1}, where

I_n : Indicator function for the event a light bulb fails & replaced on day n.

FIGURE 6.4
(a) Realization of a Bernoulli process. $I_n = 1$ indicates that a light bulb fails and is replaced in day n. (b) Realization of a binomial process. S_n denotes the number of light bulbs that have failed up to time n.



$m_I(n) = p$

$\sigma_I^2 = p(1-p)$

Find Prob. that first 4-bits are 1001:

$P[I_1 = 1, I_2 = 0, I_3 = 0, I_4 = 1] = p(1-p)(1-p)p = p^2(1-p)^2$

Sum Process:

Let $S_n = X_1 + X_2 + \ldots + X_n \qquad n = 1,2,\ldots$

$= S_{n-1} + X_n,$

pmf/pdf of S_n is found by convolution or characteristic equation methods. The block diagram shows a counting process:

FIGURE 6.6
The sum process $S_n = X_1 + \cdots + X_n$, $S_0 = 0$, can be generated in this way.

$$E[S_n] = m_S(n) = nE[X] = nm$$

$$\sigma^2_{S_n} = n\sigma^2_X = n\sigma^2$$

$$C_S(n,k) = E[(S_n - E[S_n])(S_k - E[S_k])]$$

$$= E[(S_n - nm)(S_k - km)]$$

$$= E\left[\left\{\sum_{i=1}^{n}(X_i - m)\right\}\left\{\sum_{j=1}^{k}(X_k - m)\right\}\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k} \underbrace{E[(X_i - m)(X_j - m)]}_{C_X(i,j)=\sigma^2\delta_{ij}}$$

which yields:

$$C_S(n,k) = \sum_{i=1}^{\min\{n,k\}} C_X(i,i) = \min(n,k)\sigma^2$$

FIGURE 6.7a
First-order autoregressive process.

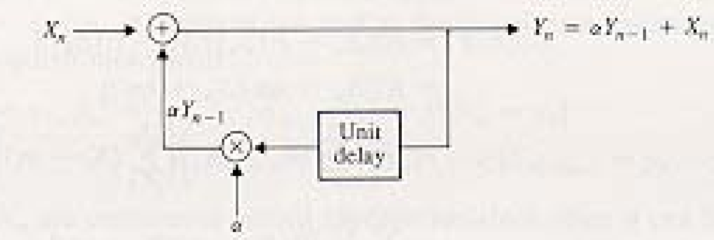
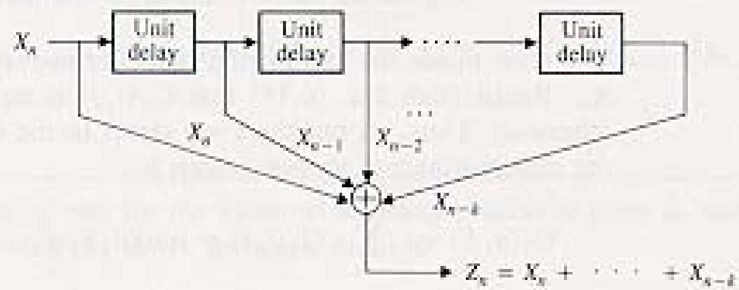


FIGURE 6.7b
Moving average process.



Both
"ARMA"
Autoregressive
Moving
Average

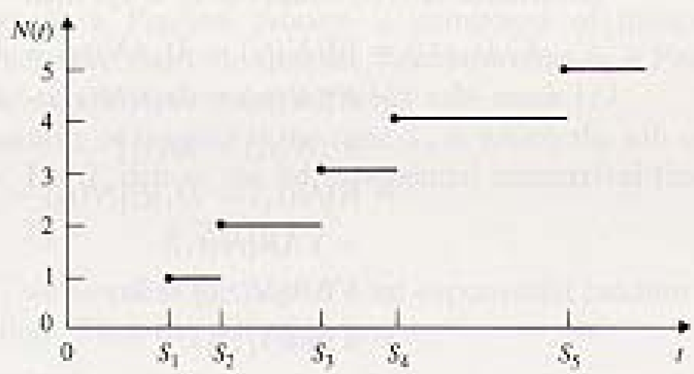
6.7a First Order autoregressive process
Linear Prediction
→ Linear estimation of α
→ Find -α
IIR or Recursive Filter
6.7b Moving Average
FIR Filter

Examples of Continuous-Time Stochastic Processes

(As a limit of Discrete-Time Stochastic Processes)

Poisson Process

FIGURE 6.8
A sample path of the Poisson counting process. The event occurrence times are denoted by $S_1, S_2, \ldots$. The jth interevent time is denoted by $X_j = S_j - S_{j-1}$.



- Events occur randomly at a rate λ

- Let N(t) be the number of occurrences in time interval [0,t]. N(t) is non-decreasing, integer-valued, continuous-time R.P.
- Let [0,t] be divided into n-intervals of duration $\delta = t/n$ and assume

1) Probability of more than one event occurring in a subinterval is negligible.
   $\Rightarrow$ **Bernoulli Trial**
2) Event occurrences in a subinterval is independent of activities in other subintervals
   $\Rightarrow$ **Bernoulli Trials are Independent**
   $\Rightarrow$ N(t) is counting process that counts number of success in n-trials. Keeping $np = \lambda t$ fixed, let $n \to \infty$ and $p \to 0$. Then we have a poisson distribution with parameter λt
   $\Rightarrow$ Poisson Process N(t) in the interval [0,t] has Poisson distribution with

$$P[N(t)=k] = \frac{(\lambda t)^2}{k!} e^{-\lambda t} \qquad \text{for } k = 0,1,2,\cdots$$

The independent and stationary increments property leads us to write for $t_1 < t_2$:

$$\begin{aligned} P[N(t_1)=i, N(t_2)=j] &= P[N(t_1)=i]P[N(t_2)-N(t_1)=j-i] \\ &= P[N(t_1)=i]P[N(t_2-t_1)=j-i] \\ &= \frac{(\lambda t_1)^i}{i!} e^{-\lambda t_1} \cdot \frac{(\lambda(t_2-t_1))^{j-i}}{(j-i)!} e^{-\lambda(t_2-t_1)} \end{aligned}$$

Autocovariance of N(t) for $t_1 < t_2$:

$$\begin{aligned} C_N(t_1,t_2) &= E[(N(t_1)-\lambda t_1)(N(t_2)-\lambda t_2)] \\ &= E[(N(t_1)-\lambda t_1)\{N(t_2)-N(t_1)-\lambda t_2+\lambda t_1+N(t_1)-\lambda t_1\}] \end{aligned}$$

$$\begin{aligned} C_N(t_1,t_2) &= \underbrace{E[(N(t_1)-\lambda t_1)]}_{0} E[(N(t_2-t_1)-\lambda(t_2-t_1))] + VAR[N(t_1)] \\ &= VAR[N(t_1)] = \lambda t_1 \qquad \text{Since } t_1 \le t_2 \end{aligned}$$

In general we have:

$$C_N(t_1,t_2) = \lambda \min\{t_1,t_2\}$$

Ex: 6.19 15 Inquires/minute; A Poisson Process Find P[N(10) = 3 and N(60) – N(45) = 2]

Poisson $\Rightarrow$ indep increment & stationary increment

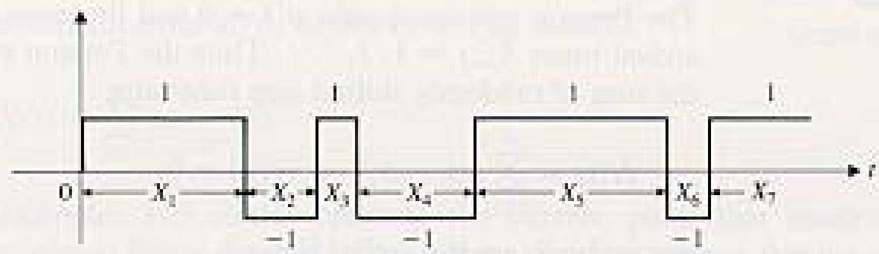
$$\begin{aligned} P[N(10)=3 \text{ and } N(60)-N(45)=2] &= P[N(10)=3]P[N(60)-N(45)=2] \\ &= P[N(10)=3]P[N(60-45)=2] \\ &= \frac{(10/4)^3 e^{-10/4}}{3!} \frac{(15/4)^2 e^{-15/4}}{2!} \end{aligned}$$

Ex: 6.22 Random Telegraph Signal

X(t) is ±1 P[X(0)= ±1]=1/2 X(t) is Poisson with rate α

Probability mass function (pmf):

FIGURE 6.9
Sample path of a random telegraph signal. The times between transitions X_j are iid exponential random variables.



$$P[X(t)=\pm 1]=P[X(t)=\pm 1\,|\,X(0)=1]P[X(0)=1]+P[X(t)=\pm 1\,|\,X(0)=-1]P[X(0)=-1]$$

Since X(t) has same polarity as X(0) only when even number of events

$$P[X(t)=\pm 1\,|\,X(0)=1]=P[N(t)=even\ \text{integer}]$$

$$=\sum_{j=0}^{\infty}\frac{(\alpha t)^{2j}}{(2j)!}e^{-\alpha t}$$

$$=e^{-\alpha t}\frac{1}{2}\left\{e^{\alpha t}+e^{-\alpha t}\right\}=\frac{1}{2}\left\{1+e^{-2\alpha t}\right\}$$

X(t) and X(0) differ in sign with odd number of events:

$$P[X(t)=\pm 1\,|\,X(0)=1]=\sum_{j=0}^{\infty}\frac{(\alpha t)^{2j+1}}{(2j+1)!}e^{-\alpha t}$$

$$=e^{-\alpha t}\frac{1}{2}\left\{e^{\alpha t}+-e^{-\alpha t}\right\}=\frac{1}{2}\left\{1-e^{-2\alpha t}\right\}$$

Therefore,

$$P[X(t)=1]=\frac{1}{2}\cdot\frac{1}{2}\left\{1+e^{-2\alpha t}\right\}+\frac{1}{2}\cdot\frac{1}{2}\left\{1-e^{-2\alpha t}\right\}=\frac{1}{2}$$

$$P[X(t)=-1]=1-P[X(t)=1]=\frac{1}{2}$$

Thus signal is equally likely to be ± 1. Next we find the mean, variance and autocovariance functions.

$$m_X(t)=(1)\cdot P[X(t)=1]+(-1)\cdot P[X(t)=-1]=0$$

$$VAR[X(t)]=E\left[X(t)^2\right]=(1)^2\cdot P[X(t)=1]+(-1)^2\cdot P[X(t)=-1]=1$$

$$C_X(t_1,t_2)=E[X(t_1)X(t_2)]=(1)P[X(t_1)=X(t_2)]+(-1)P[X(t_1)\neq X(t_2)]$$

$$=\frac{1}{2}\left\{1+e^{-2\alpha|t_2-t_1|}\right\}$$

Note: Time samples of X(t) become less correlated as time between them increases. Also it does not matter which time is greater.

Ex: 6.23 Filtered Poisson Impulse Train: Zero at t = 0 and increases by one unit at random arrival times: S_j , i = 1,2,…

$$N(t) = \sum_{i=1}^{\infty} u(t - S_i) \qquad N(0) = 0$$

We can view N(t) as the integral of a train of delta functions

$$Z(t) = \sum_{i=1}^{\infty} \delta(t - S_i)$$

We can obtain other continuous-time processes by replacing the step function by another function h(t)—Figure 6.10b.

FIGURE 6.10a
Poisson process as integral of train of delta functions.

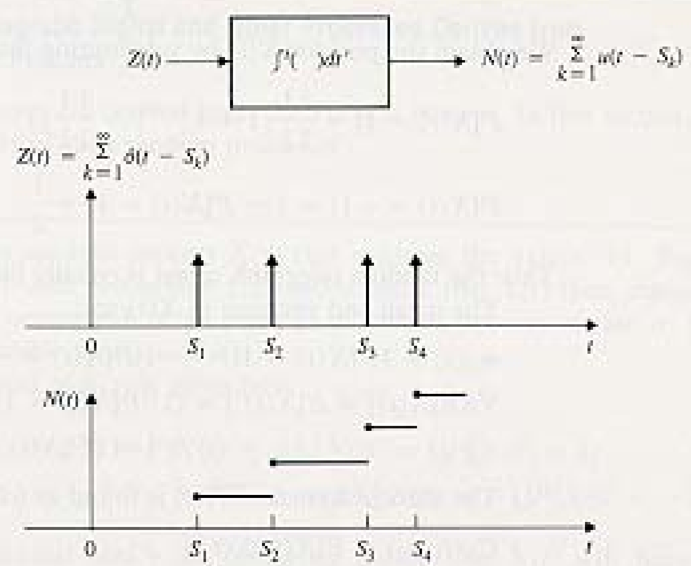
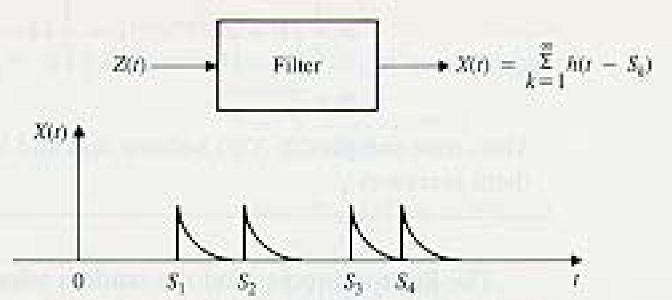


FIGURE 6.10b
Filtered train of delta functions.



Ex: 6.24 Shot Noise: h(t) is the current pulse generator when a photoelectron hits a detector.

$$X(t) = \sum_{i=1}^{\infty} h(t - S_i)$$

Find expected value: $E[X(t)] = E[E[X(t) \mid N(t)]]$, where N(t) is number of impulses that occurred up to time t

$$E[E[X(t) \mid N(t) = k]] = E\left[\sum_{j=1}^{\infty} h(t - S_j)\right] = \sum_{j=1}^{\infty} E[h(t - S_j)]$$

Since independent and uniformly distributed in interval [0,t]:

$$E[h(t - S_j)] = \int_0^t h(t - s)\frac{ds}{t} = \frac{1}{t}\int_0^t h(u)du$$

Thus:

$$E[X(t) \mid N(t) = k] = \frac{k}{t}\int_0^t h(u)du$$

and

$$E[X(t) \mid N(t)] = \frac{N(t)}{t}\int_0^t h(u)du$$

Finally, we obtain:

$$E[X(t)] = E[E[X(t) \mid N(t)]] = \frac{E[N(t)]}{t}\int_0^t h(u)du$$

$$= \lambda \int_0^t h(u)du \qquad where \quad E[N(t)] = \lambda t$$

The integral is finite, as t becomes large E[N(t)] $\rightarrow$ constant

(Skip Wiener Process and Brownian Motion)

Stationary Random Process (Strictly Stationary)

- Nature of randomness stays unchanged with time (Independent of time origin).
- A discrete-time or continuous S.P. X(t) is stationary if the joint distribution of any set of samples does not depend on the time origin:

$$F_{X(t_1)\cdots X(t_k)}(x_1,\ldots,x_k) = F_{X(t_1+\tau)\cdots X(t_k+\tau)}(x_1,\ldots,x_k)$$

for all τ , all k, and all choices of $t_1, \ldots, t_k$

- First-order cdf of a stationary R.P. must be independent of t.

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x) \qquad \forall t, \forall \tau$$
$$m_{X(t)} = E[X(t)] = m \qquad \forall t$$
$$VAR[X(t)] = \sigma^2 \qquad \forall t$$

- 2^{nd} order cdf of a stationary R.P. can depend only on the time difference between the samples:

$$F_{X(t_1)X(t_2)}(x_1, x_2) = F_{X(t_1)X(t_2 - t_1)}(x_1, x_2) \qquad \forall t_1, t_2$$
$$R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau) \qquad where\ \tau = t_2 - t$$
$$C_X(t_1, t_2) = C_X(t_2 - t_1) = C_X(\tau) \qquad where\ \tau = t_2 - t$$

Ex: 6.26 Show i.i.d. R.P. is stationary:

$$F_{X(t_1)\cdots X(t_k)}(x_1, x_2, \ldots x_k) = F_X(x_1) F_X(x_2) \cdots F_X(x_k)$$
$$= F_{X(t_1+\tau)\cdots X(t_k+\tau)}(x_1, \ldots x_k)$$

for all k, t_1, … , t_k.

Therefore, i.i.d. R.P. is stationary.

Ex: 6.27 Is sum process a discrete-time stationary process?

$S_n = X_1 + X_2 + \ldots + X_n$ where X_i are iid sequences
$m_S(n) = nm$ $VAR[S_n] = n\sigma^2$

Mean and Variance are not constant but linear with time index n, thus sum process cannot be a stationary process.

Ex: 6.28 Show Random Telegraph Signal of Ex: 6.22 is stationary.

Need to show that:

$$P[X(t_1) = a_1, \ldots, X(t_k) = a_K] = P[X(t_1 + \tau) = a_1, \ldots, X(t_k + \tau) = a_K]$$

for any k, any $t_1 < \cdots < t_k$, and $a_j = \pm 1$.

Since the Poisson process has the independent increments property:

$$P[X(t_1) = a_1, \ldots, X(t_k) = a_K] = P[X(t_1) = a_1] P[X(t_2) = a_2 \mid X(t_1) = a_1] \cdots$$
$$P[X(t_k) = a_k \mid X(t_{k-1}) = a_{k-1}]$$

Since the values of the random telegraph at t_1, … , t_k is determined by time intervals (t_j, t_{j+1}):

$$P\left[X(t_1+\tau)=a_1,\ldots,X(t_k+\tau)=a_K\right]$$
$$=P\left[X(t_1+\tau)=a_1\right]P\left[X(t_2+\tau)=a_2\mid X(t_1+\tau)=a_1\right]\cdots$$
$$P\left[X(t_k+\tau)=a_k\mid X(t_{k-1}+\tau)=a_{k-1}\right]$$

The transition probabilities in the above two equations are equal since

$$P\left[X(t_{j+1})=a_{j+1}\mid X(t_j)=a_j\right]$$
$$=\begin{cases}\frac{1}{2}\{1+e^{-2\alpha(t_{j+1}-t_j)}\} & if\ a_j=a_{j+1}\\ \frac{1}{2}\{1-e^{-2\alpha(t_{j+1}-t_j)}\} & if\ a_j\neq a_{j+1}\end{cases}$$
$$=P\left[X(t_{j+1}+\tau)=a_{j+1}\mid X(t_j+\tau)=a_j\right]$$

Thus they differ only in the first term

$$P\left[X(t_1)=a_1\right]\quad and\quad P\left[X(t_1+\tau)=a_1\right]$$

if $P\left[X(0)=\pm1\right]=1/2$

then:

$$P\left[X(t_1)=a_1\right]=1/2,\ P\left[X(t_1+\tau)=a_1\right]=1/2$$

Therefore,

$$P\left[X(t_1)=a_1,\ldots,X(t_k)=a_K\right]=P\left[X(t_1+\tau)=a_1,\ldots,X(t_k+\tau)=a_K\right]$$

The process is stationary.

If $P\left[X(0)=\pm1\right]\neq1/2$ they are not equal.

However,

$$P\left[X(t)=a\right]=P\left[X(t)=a\mid X(0)=a1\right]$$
$$=\begin{cases}\frac{1}{2}\left\{1+e^{-2\alpha t}\right\} & if\ a=1\\ \frac{1}{2}\left\{1-e^{-2\alpha t}\right\} & if\ a=-1\end{cases}$$

for small t , X(t) is close to 1; but as t increases X(t) = 1 $\Rightarrow$ ½
thus as t becomes large the joint pmf's become equal. Therefore when the process settles down into "steady state" is becomes stationary.

Wide-Sense Stationary Random Processes

A discrete-time or continuous-time random process X(t) is **wide-sense stationary (WSS)** if

$m_X(t) = m$ for all t,

and

$C_X(t_1, t_2) = C_X(t_1 - t_2)$ for all t_1, t_2

X(t) and Y(t) are **jointly wide-sense stationary** if they are both wide-sense stationary and if their cross-covariance depends only on $t_1 - t_2$

$C_{XY}(t_1, t_2) = C_{XY}(\tau)$ and $R_{XY}(t_1, t_2) = R_{XY}(\tau)$ $\tau = t_2 - t_1$

All stationary random processes are wide-sense stationary.

Ex: 6.29 X_n : Two interleaved sequences of indep. random variables.

For n even $X_n = \pm 1$ p =1/2

For n odd X_n = 1/3, -3 p = 9/10 and 1/10

$$m_X(n) = 0 \qquad \textit{for all } n$$

$$C_X(i, j) = \begin{cases} E[X_i]E[X_j] = 0 & i \neq j \\ E[X_i^2] = 1 & i = j \end{cases}$$

Therefore, X_n is wide-sense stationary.

Properties of WSS processes:

1. Autocorrelation function at $\tau = 0$ $\Rightarrow$ **average power**

$$R_X(0) = E\left[X(t)^2\right] \qquad \text{for all t}$$

2. Autocorrelation function is an even function of τ:

$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t-\tau)] = R_X(-\tau)$$

3. Autocorrelation function is a measure of the rate of change of random processes:

$$P\left[|X(t+\tau) - X(t)| > \varepsilon\right] = P\left[(X(t+\tau) - X(t))^2 > \varepsilon^2\right]$$
$$\leq \frac{E\left[(X(t+\tau) - X(t))^2\right]}{\varepsilon^2}$$
$$\leq \frac{2\{R_X(0) - R_X(\tau)\}}{\varepsilon^2}$$

4. Autocorrelation function is maximum at τ = 0. Because,

$$E[XY]^2 \le E[X^2].E[Y^2]$$
$$R_X(\tau)^2 = E[X(t+\tau)X(t)]^2 \le E[X^2(t+\tau)].E[X^2(t)] = R_X(0)^2$$

5. If $R_X(0) = R_X(d)$ then $R_X(\tau)$ is periodic with period *d* and X(t) is **mean-square periodic** i.e. $E\left[(X(t+d) - X(t))^2\right] = 0$

6. $R_X(\tau)$ approaches the square of the mean of X(t) as $\tau \to \infty$

Let X(t) = m + N(t), where N(t) is a zero-mean process for which

$R_X(\tau) \to 0$ as $\tau \to \infty$, then

$$R_X(\tau) = E[(m + N(t+\tau))(m + N(t))] = m^2 + 2mE[N(t)] + R_N(\tau)$$
$$= m^2 + R_N(\tau) \to m^2 \qquad as\ \tau \to \infty$$

Ex: 6.30

Fig 6.12a is autocorrelation function for random telegraph signal

$$R_X(\tau) = e^{-2\alpha|\tau|}$$

Fig 6.12b is the autocorrelation function for a sinusoid

$$R_X(\tau) = \frac{a^2}{2}\cos(2\pi\, f_0\tau)$$

Fig 6.12c is autocorrelation function for the process

Z(t) = X(t) + Y(t) + m

Where X(t) is random telegraph process, Y(t) is sinusoid with random phase, and m is constant. X(t) and Y(t) are independent.

$$R_Z(\tau) = E[\{X(t+\tau) + Y(t+\tau) + m\}\{X(t) + Y(t) + m\}]$$
$$= R_X(\tau) + R_Y(\tau) + m^2$$

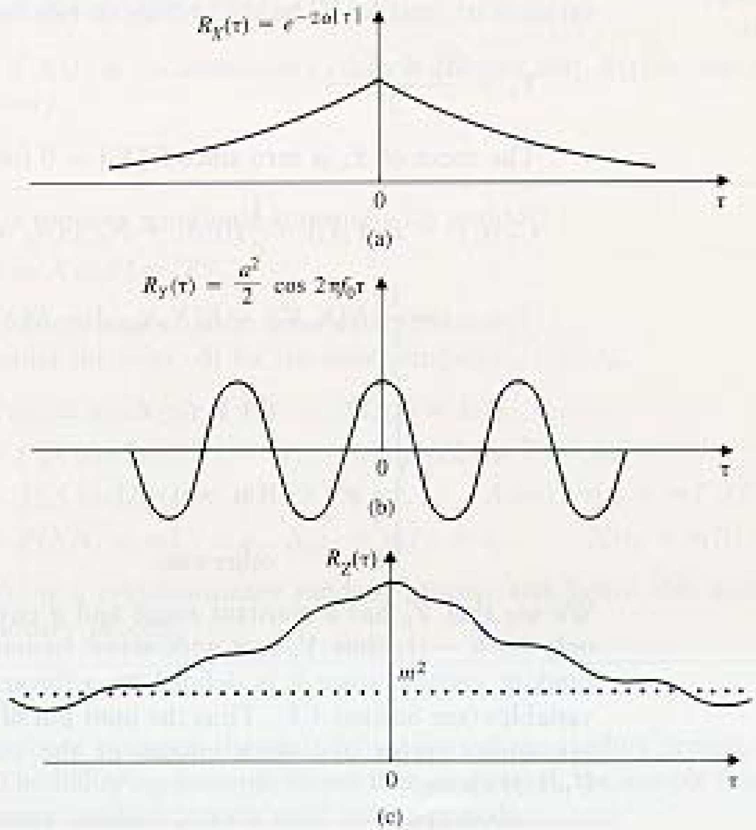


FIGURE 6.12
(a) Autocorrelation function of a random telegraph signal. (b) Autocorrelation function of a sinusoid with random phase. (c) Autocorrelation function of a random process that has nonzero mean, a periodic component, and a "random" component.

(Skip Wide-Sense Stationary Gaussian Random Processes)
(Skip Cyclostationary Random Processes, Skip Section 6.6)

Time Averages of Random Processes and Ergodic Theorems

Sometimes we are interested in estimating the mean or autocorrelation functions from the **time average** of a single realization

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,\xi)dt$$

and

$$VAR\left[\langle X(t)\rangle_T\right] = \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)C_X(u)du$$

where $u = t - t' \;\; for \; -2T < u < 2T$

Let X(t) be a wide-sense stationary (WSS) process with $m_X(t) = m$, then

$\lim_{T\to\infty}\langle X(t)\rangle_T = m$ in the mean square sense, if and only if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)C_X(u)du = 0$$

A WSS process is said to be **mean ergodic** if it satisfies the above conditions.
A time-average estimate for the autocorrelation function of Y(t) is

$$\langle Y(t+\tau)Y(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} Y(t+\tau)Y(t)dt$$

The time-average autocorrelation converges to $R_Y(\tau)$ in the mean square sense if *Y(t)* is mean ergodic.

For discrete case, the mean and autocorrelation functions of X_n are:

$$\langle X_n\rangle_T = \frac{1}{2T+1}\sum_{n=-T}^{T} X_n$$

$$\langle X_{n+k}X_n\rangle_T = \frac{1}{2T+1}\sum_{n=-T}^{T} X_{n+k}X_n$$

If X_n is WSS, then

$$E[\langle X_n\rangle_T] = m \quad \text{and} \quad VAR[\langle X_n\rangle_T] = \frac{1}{2T+1}\sum_{k=-2T}^{2T}\left(1-\frac{|k|}{2T+1}\right)C_X(k)$$

$[\langle X_n\rangle_T]$ is mean ergodic if $VAR[\langle X_n\rangle_T]$ approaches zero with increasing T.

Ex: 6.43 Random Telegraph Process

$$C_X(\tau) = e^{-2\alpha|\tau|}$$

$$VAR[\langle X(t)\rangle_T] = \frac{1}{2T}\int_0^{2T}\left(1-\frac{u}{2T}\right)e^{-2\alpha u}du < \frac{1}{2T}\int_0^{2T} e^{-2\alpha u}du = \frac{1-e^{-4\alpha T}}{2\alpha T}$$

as $T \to \infty$ $VAR[\langle X(t)\rangle_T] \to 0$, thus process is **mean ergodic.**

#6.3 Fair coin toss Heads $X_n = (-1)^n$ Tails $X_n = (-1)^{n+1}$

a) Sketch

		n=0	n=1	n=2	…	
If Heads	X_n	1	-1	1	-1	…
If Tails	X_n	-1	1	-1	1	…

b) Find the pmf

n even $P[X_n = 1] = P[\text{Heads}] = 1/2$

n odd $P[X_n = -1] = P[\text{Tails}] = 1/2$

c) Find the joint pmf

k even

$$P[X_n = 1, X_{n+k} = 1] = P[\text{Heads}] = 1/2$$

$P[X_n = -1, X_{n+k} = -1] = P[\text{Tails}] = 1/2$
$P[X_n = \pm 1, X_{n+k} = \mp 1] = 0$

k odd

$P[X_n = 1, X_{n+k} = -1] = P[\text{Heads}] = 1/2$
$P[X_n = -1, X_{n+k} = 1] = P[\text{Tails}] = 1/2$
$P[X_n = \pm 1, X_{n+k} = \pm 1] = 0$

d) Find the mean and autocovariance

$E[X_n] = 1(1/2) + (-1)(1/2) = 0$

k even $\quad E[X_n X_{n+k}] = (1)^2(1/2) + (-1)^2(1/2) = 1$

k odd $\quad E[X_n X_{n+k}] = (1)(-1)(1/2) + (-1)(1)(1/2) = -1$

#6.15 $\quad Z(t) = Xt + Y \quad m_X, m_Y, \sigma_X^2, \sigma_Y^2, \rho_{XY}$

a) Find mean and autocovariance of $Z(t)$

$$E[Z(t)] = E[Xt + Y] = E[X]t + E[Y] = tm_X + m_Y = m_Z$$

$$C_Z(t_1,t_2) = E[(Xt_1 + Y)(Xt_2 + Y)] - m_Z(t_1)m_Z(t_2)$$

$$= t_1t_2E[X^2] + (t_1. + t_2)E[XY] + E[Y^2]$$

$$- t_1t_2m_X^2 - (t_1. + t_2)m_X m_Y - m_Y^2$$

$$= t_1t_2\sigma_X^2 + (t_1. + t_2)\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2$$

b) Find pdf of $Z(t)$ if X and Y are jointly Gaussian r.v.

From example 4.32, (Page:222), where Z=X+Y

$$f_{Z(t)}(z) = \frac{\exp\left\{-\dfrac{(z - tm_X - m_y)^2}{2(t^2\sigma_X^2 + 2t\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2)}\right\}}{\sqrt{2\pi\left(t^2\sigma_X^2 + 2t\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2\right)}}$$

#6.53 $\quad X(t) = A\cos wt + B\sin wt \quad$ A, B iid, zero mean

a) Show $X(t)$ is WSS

$$E[X(t)] = E[A\cos wt + B\sin wt]$$

$$= E[A]\cos wt + E[B]\sin wt = 0$$

$$C_X(t_1,t_2) = E[(A\cos wt_1 + B\sin wt_1)(A\cos wt_2 + B\sin wt_2)]$$

$$C_X(t_1,t_2) = E[A^2]\cos wt_1 \cos wt_2 + E[B^2]\sin wt_1 \sin wt_2$$
$$+ E[A]E[B]\cos wt_1 \sin wt_2 + E[A]E[B]\sin wt_1 \cos wt_2$$
$$= E[A^2]\cos wt_1 \cos wt_2 + E[B^2]\sin wt_1 \sin wt_2$$
$$= E[A^2]\underbrace{\{\cos wt_1 \cos wt_2 + \sin wt_1 \sin wt_2\}}_{\frac{1}{2}\cos w(t_1 - t_2)}$$

where we assumed $E[A^2] = E[B^2]$

$$= \frac{1}{2}E[A^2]\cos w(t_1 - t_2) = \frac{1}{2}E[A^2]\cos w\tau$$

$\therefore$ **X(t) is WSS**

b) Show X(t) is not strictly-stationary

$$E\left[X^3(t)\right] = E\left[(A\cos wt + B\sin wt)^3\right]$$
$$= E[A^3\cos^3 wt + 3A^2B\cos^2 wt\sin wt + 3AB^2\cos wt\sin^2 wt$$
$$+ B^2\sin^3 wt]$$
$$= E\left[A^3\right]\cos^3 wt + E\left[B^3\right]\sin^3 wt = E\left[A^3\right]\left(\cos^3 wt + \sin^3 wt\right)$$
$$= \frac{E\left[A^3\right]}{4}\underbrace{\{3(\cos wt + \sin wt) + (\cos 3wt - \sin 3wt)\}}_{\text{these terms depend on } t \text{ explicitly}}$$

moment of X(t) depends explicitly on time-origin

$\Rightarrow$ X(t) is not strictly-stationary

#6.78 Find variance of Example 6.42 page 379.

X(t) = A A is zero mean, unit-variance r.v.

$$E[X(t)] = E[A] = 0$$
$$E[X(t_1)X(t_2)] = E\left[A^2\right] = 1$$

$$VAR\left[\langle X(t)\rangle_T\right] = \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|u|}{2T}\right)C_X(u)du = 2.\frac{1}{2T}\int_0^{2T}\left(1 - \frac{u}{2T}\right)du = 1$$

$\Rightarrow$ This process is <u>not</u> mean-ergodic