Chapter 5: Sums Random Variables

Let $S_n = X_1 + X_2 + \ldots + X_n$

$$E[S_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

Ex: 5.1 Find σ_Z^2 where $Z = X + Y$ $\quad E[Z] = E[X] + E[Y]$

$$\sigma_Z{}^2 = E\left\{\left[(X+Y)-(E[X]+E[Y])\right]^2\right\} = E\left\{\left[(X-E[X])+(Y-E[Y])\right]^2\right\}$$
$$= E\left[(X-E[X])^2\right] + E\left[(Y-E[Y])^2\right] + 2E\left[(X-E[X])(Y-E[Y])\right]$$
$$= \sigma_X{}^2 + \sigma_Y{}^2 + 2COV(X,Y)$$

If X and Y are uncorrelated or independent, then $COV(X,Y) = 0$ and

$$\sigma_Z{}^2 = \sigma_X{}^2 + \sigma_Y{}^2$$

$$\sigma^2{}_{S_n} = E\left\{\sum_{j=1}^{n}\left(X_j - E[X_j]\right)\sum_{k=1}^{n}\left(X_k - E[X_k]\right)\right\}$$

$$= \sum_{k=1}^{n}\sigma^2{}_{X_k} + \underbrace{\sum_{\substack{j=1 \\ but\ j\neq k}}^{n}\sum_{k=1}^{n} COV(X_j.X_k)}_{\mathbf{0}} = \sum_{k=1}^{n}\sigma_{X_k}{}^2$$

Ex: 5.2 X and Y are i.i.d. r.v. with μ and σ^2

$$E[S_n] = E[X_1] + \ldots + E[X_n] = n\mu$$
$$\sigma_{S_n}{}^2 = n\sigma_{X_j}{}^2 = n\sigma^2$$

pdf of Sums of r.v.:

$n = 2$ $\quad Z = X + Y$ and X & Y are independent.

Let us use characteristic function approach:

$$\Phi_Z(w) = E\left[e^{-jwZ}\right] = E\left[e^{-jw(X+Y)}\right] = E\left[e^{-jwX}\right]\cdot E\left[e^{-jwY}\right]$$
$$= \Phi_X(w)\cdot\Phi_Y(w)$$

Since $\Phi_Z(w) \Leftrightarrow f_Z(z)$

we can write equivalently:

$$f_Z(z) = f_X(x) * f_Y(y)$$

and

$$S_n = X_1 + \ldots + X_n \quad \Rightarrow \quad \Phi_{X_1}(w)\cdot\Phi_{X_2}(w)\cdots\Phi_{X_n}(w)$$

On the other hand, if $\{X_i\}$ are all integer-valued r.v., then we can use probability generating function approach:

$$G_N(z) = E\left[z^N\right] \quad \text{and} \quad N = X_1 + \ldots + X_n$$

which leads to:

$$G_N(z) = E\left[z^{X_1 + \ldots + X_n}\right] = G_{X_1}(z) \cdot G_{X_2}(z) \cdots G_{X_n}$$

Ex: 5.4 and 5.5 Let $S_n = X_1 + \ldots + X_n$ be sum of i.i.d. with

$$\Phi_{X_k}(w) = \Phi_X(w) \quad k = 1,2,\ldots n$$

then

$$\Phi_{S_n}(w) = \{\Phi_X(w)\}^n$$

pdf of S_n if X_k are i.i.d. exponential r.v.

$$\Phi_X(w) = \frac{\alpha}{\alpha - jw}$$

then

$$\Phi_{S_n}(w) = \left[\frac{\alpha}{\alpha - jw}\right]^n$$

⇒ S_n : m-Erlang r.v. of Table 3.2

Sample Mean, M_n

Let $X_1, \ldots, X_n$ be n independent outcomes from experiments with an unknown mean, μ. Since they are from the same population X_i is i.i.d. with the same pdf:

$$M_n = \frac{1}{n}\sum_{j=1}^{n} X_j \qquad \Rightarrow \textit{Centroid, Center of Gravity}$$

$$E[M_n] = E\left[\frac{1}{n}\sum_{j=1}^{n} X_j\right] = \frac{1}{n}\sum_{j=1}^{n} E[X_j] = \frac{n}{n}E[X_j] = \mu$$

$$\sigma_{M_n}{}^2 = E\left[(M_n - \mu)^2\right] = E\left[(M_n - E[M_n])^2\right]$$

But

$$S_n = X_1 + X_2 + \ldots X_n \Rightarrow M_n = \frac{S_n}{n}$$

$$\sigma_{S_n}{}^2 = n\sigma_{X_j}{}^2 = n\sigma_X{}^2$$

Then:

$$\sigma_{M_n}{}^2 = \frac{1}{n^2}\sigma_{S_n}{}^2 = \frac{1}{n}\sigma_X{}^2$$

Chebyshev's Inequality for M_n (Sample Mean):

$$P\left[\,|M_n - E[M_n]| \geq \varepsilon\right] \leq \frac{\sigma_{M_n}^{\;2}}{\varepsilon^2} = \frac{(1/n).\sigma^2}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

and compliment:

$$P\left[\,|M_n - E[M_n]| < \varepsilon\right] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

Ex: 5.9 Given noisy voltage measurement with:

$$X_j = v + N_j \quad with \quad N_j : N(0, 1\mu V)$$

How many measurements are needed (n = ?) for M_n to be within $\varepsilon = 1\ \mu V$ of true mean is at least .99?

$$P\left[|M_n - \mu| < \varepsilon\right] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

$$= 1 - \frac{(1\mu V)^2}{n(1\mu V)^2} = 1 - \frac{1}{n} = 0.99 \qquad \Rightarrow n = 100$$

Weak-Law of Large Numbers:

Let $X_1, X_2, \ldots$ be a sequence of iid R.V. with $E[X_i] = \mu$, then for $\varepsilon > 0$

$$\lim_{n\to\infty} P\left[\,|M_n - \mu| < \varepsilon\right] = 1$$

>>>(seeFig 5.1 p 278 for interpretation)

Strong-Law of Large Numbers:

Let $X_1, X_2, \ldots$ be a sequence of iid R.V. with $E[X_i] = \mu$ and finite variance, then

$$P\left[\lim_{n\to\infty} M_n = \mu\right] = 1$$

Ex: 5.10 Bernoulli trials with unknown $\mu = p$ and $\sigma^2 = p(1-p)$
How large n should be to have 0.95 probability that $f_A(n)$ is within 1% of $p = P[A]$?

If $X = I_A$ indicator function, then:

$$E[X] = E[I_A] = \mu = p$$

and

$$\sigma_{I_A}^{\;2} = p(1-p)$$

$$\frac{d\sigma^2}{dp} = 1 - 2p = 0 \Rightarrow p^* = \frac{1}{2}$$

$$\Rightarrow \sigma_{I_A}{}^2 \le \left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

$$\therefore\ P\left[\left|f_A(n) - p\right| \le \varepsilon\right] \le \frac{\sigma^2}{n\varepsilon^2} \le \frac{1/4}{n\varepsilon^2}$$

Note: Chebyshev inequality results in loose bounds.

Since: $\varepsilon = 1\% = 0.01\ and\ 1 - 0.95 = \dfrac{1}{4n(0.01)^2} \Rightarrow n \ge 50{,}000$

Central Limit Theorem:

Let $X_1, X_2, \ldots$ be a sequence of iid RV. With μ and σ^2 and

$S_n = X_1 + X_2 + \ldots + X_n$

Let us define a new r.v.: $Z_n = \dfrac{S_n - n\mu}{\sigma\sqrt{n}}$,

Then

$$\lim_{n\to\infty} P[Z_n \le z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$$

⇒ Sum of iid R.V. with any distribution in the limit approaches to that of Gaussian statistics!

Ex: 5.11 Orders: iid with $\mu = \$8$ $\sigma = \$2$

a.) Estimate probability that first 100 customers will spend ≥ \$840.

$S_{100} = X_1 + X_2 + \ldots + X_{100}$

$E[S_{100}] = n\mu = 800$

$$Z_{100} = \frac{S_{100} - 800}{2\sqrt{100}} = \frac{S_{100} - 800}{20}$$

$$\sigma_{S_{100}}{}^2 = n\sigma^2$$
$$= 100x4 = 400$$

From Figure 5.5 in Page 284 and Table 3.3 we have

$$P[S_{100} > \$840] = P\left[Z_{100} > \frac{840 - 800}{20}\right] = P[Z_{100} > 2] \approx Q(2) = 0.0228$$

b.) Prob. that 100 customers will spend \$780 ≤ S_{100} ≤ \$820?

$$P[780 \le S_{100} \le 820] = P[-1 \le Z_{100} \le 1]$$
$$\approx 1 - 2Q(1) = 0.682$$

Ex: 5.12 After how many orders 90% sure that total expenditure > \$1000?
Find n for which $P[S_n > \$1000] = 0.90$

Recall $E[S_n] = 8n$ $\sigma_S^2 = 4n$

$$P[S_n > \$1000] = P\left[Z_n > \frac{1000-8n}{2\sqrt{n}}\right] = 0.90$$

Note: $Q(-X) = 1\text{-}Q(X) \Rightarrow 0.90$
$\Rightarrow Q(-X) = 0.1 \Rightarrow X = -1.2815$

From Table 3.4, we get: $\frac{1000-8n}{2\sqrt{n}} = -1.2815$

$$\Rightarrow \quad 8n - 1.2815(2)\sqrt{n} - 1000 = 0$$

$$\Rightarrow \quad \sqrt{n} = 11.34 \qquad \Rightarrow n = 128.6 \; or \; 129$$

Gaussian Approximation to Binomial Probability:
From Central Limit Theorem for n large:

$$P[X = k] \approx P\left[k - \frac{1}{2} < Y < k + \frac{1}{2}\right] \approx \frac{\exp\left\{-\frac{(k-np)^2}{2np(1-p)}\right\}}{\sqrt{2\pi np(1-p)}}$$

where $\mu = np$ and $\sigma^2 = np(1-p)$ of binomial distribution.

Ex: 5.14 In Ex: 5.10 Using Strong Law of Large Numbers we have obtained:

$\Rightarrow$ n $\geq$ 50,000

Let $f_A(n)$ be relative frequency of A in n-Bernoulli trials and let us use the Gaussian approximation to Binomial distribution:

$$E[f_A(n)] = p \text{ and } \sigma_A^2 = \frac{p(1-p)}{n}$$

$$\Rightarrow \quad Z_n = \frac{f_A(n) - p}{\sqrt{\frac{p(1-p)}{n}}} \quad \text{with } E[Z_n] = 0 \text{ and } \sigma_{Z_n}^2 = 1 \text{ if } n \text{ is large.}$$

then,

$$P\big[|f_A(n) - p| < \varepsilon\big] \approx P\left[|Z_n| < \frac{\varepsilon\sqrt{n}}{\sqrt{p(1-p)}}\right] = 1 - 2Q\left(\frac{\varepsilon\sqrt{n}}{\sqrt{p(1-p)}}\right)$$

using

$$\frac{d\sigma^2}{dp} = 0 \quad \Rightarrow \frac{d\ p(1-p)}{dp} = 0 \quad \Rightarrow p^* = \frac{1}{2} \quad \Rightarrow p(1-p) \le \frac{1}{4}$$

then

$$\sqrt{p(1-p)} \le \sqrt{\frac{1}{4}} = \frac{1}{2}$$

which results in:

$$P\left[\,\left|f_A(n) - p\right| < \varepsilon\right] > 1 - 2Q\left(2\varepsilon\sqrt{n}\right)$$

$$0.95 \text{ is required} \Rightarrow 2Q\left(2\varepsilon\sqrt{n}\right) = \frac{1-0.95}{2} = 0.025$$

from Table 3.3

$\Rightarrow\ 2\varepsilon\sqrt{n} \approx 1.95 \ \Rightarrow$ n $\ge$ 9506; Much smaller than the result in Ex: 5.10

(Skip Sections 5.4, 5.5 & 5.6)

Finding Distributions Using DFT(FFT)

Let X be an integer-valued discrete R.V. in the range: *{0, 1, …, N-1}*: then

$$\Phi_X(w) = \sum_{k=0}^{N-1} p_k\, e^{jwk}$$

where $p_k = P[X = k]$ is pmf

and the characteristic function: $\Phi_X(w)$ is periodic in 2π since:

$$e^{jwk} = e^{jwk}\, e^{j2\pi k} = e^{jk(w+2\pi)}$$

Let us sample function: $\Phi_X(w)$ at *N-equally spaced* values:

$$c_m = \Phi_X\left(\frac{2\pi}{N}m\right) = \sum p_k\, e^{j\frac{2\pi\, k\, m}{N}} \quad \text{for m=0,1,…,N-1}$$

Inverse DFT would yield:

$$p_k = \frac{1}{N}\sum c_m\, e^{-j\frac{2\pi\, k\, m}{N}} \qquad \text{for k=0,1,…,N-1}$$

Extend the range of X to *{0, 1, …, N-1, N, …, L-1}* by defining

$$p'_j = \begin{cases} p_j & 0 \le j \le N-1 \\ 0 & N \le j \le L-1 \end{cases}$$

DFT yields:

$$c_m = \Phi_X\left(\frac{2\pi}{L}m\right) \qquad \textit{for } m = 0, 1, \ldots, L\text{-}1$$

Sum of iid integers: $Z = X_1 + X_2 + \dots + X_n$

If $X_i : \{0, 1, \dots, N\text{-}1\}$ then $Z : \{0, \dots, n(N\text{-}1)\}$

Obtain pmf of Z from DFT evaluated at $L = n(N-1) + 1$ *points*

$$d_m = \Phi_Z\left(\frac{2\pi m}{L}\right) = \left[\Phi_X\left(\frac{2\pi m}{L}\right)\right]^n \qquad \text{for m = 0, 1, …, L-1}$$

Since

$$\Phi_Z(w) = [\Phi_X(w)]^n$$

then

$$P[Z = k] = \frac{1}{L}\sum_{m=0}^{L-1} d_m\, e^{-j\frac{2\pi mk}{L}} \qquad \textit{for } k = 0, 1, \dots, L\text{-}1$$

Ex: 5.33 Let $Z = X_1 + X_2$ with $\Phi_X(w) = \frac{1}{3} + \frac{2}{3}e^{jw}$

Find: P[Z=1] via DFT. Since X: $\{0,1\}$, then Z: $\{0,1,2\}$

$$\Phi_Z(w) = [\Phi_X(w)]^2$$

$$d_m = \Phi_Z(w) = [\Phi_X(w)]^2 = \left[\frac{1}{3} + \frac{2}{3}e^{j\frac{2\pi m}{3}}\right]^2 \qquad \textit{for } m = 0,1,2$$

$$d_0 = \left[\frac{1}{3} + \frac{2}{3}\right]^2 = 1 \qquad d_1 = \left[\frac{1}{3} + \frac{2}{3}e^{j\frac{2\pi}{3}}\right]^2 = \frac{1}{9} + \frac{4}{9}e^{j\frac{2\pi}{3}} + \frac{4}{9}e^{j\frac{4\pi}{3}}$$

$$d_1 = \frac{1}{9} + \frac{4}{9}\cos(120) + \frac{4}{9}j\sin(120) + \frac{4}{9}\cos(240) + \frac{4}{9}j\sin(240) = -\frac{1}{3}$$

Similarly,

$$d_2 = d_1^* = -1/3$$

Substituting these in pmf equations:

$$P[Z = 1] = \frac{1}{3}\left\{d_0 + d_1 e^{-j\frac{2\pi}{3}} + d_2 e^{-j\frac{4\pi}{3}}\right\} = \frac{1}{3}\left\{1 - \frac{1}{3}\left(e^{-j\frac{2\pi}{3}} + e^{-j\frac{4\pi}{3}}\right)\right\} = \frac{4}{9}$$

Let $S_X = \{0, 1, 2, \dots\}$ be an open-ended sequence and $\Phi_X(w)$ is known. We want to obtain pmf values p'_k from a finite set of samples the characteristic function:

$$p'_k = \frac{1}{N}\sum_{m=0}^{N-1} c_m\, e^{-j\frac{2\pi km}{N}} \qquad \textit{for } k = 0, 1, \dots, N\text{-}1$$

and

$$c_m = \Phi_X\left(\frac{2\pi m}{N}\right) \qquad \textit{for } m = 0, 1, \ldots, N\text{-}1$$

$$c_m = \; = \sum_{n=0}^{\infty} p_n e^{j\frac{2\pi mn}{N}}$$

$$= (p_0 + p_N + p_{2N} + \ldots)e^{j0} + (p_1 + p_{N+1} + \ldots)e^{j\frac{2\pi m}{N}}$$

$$+ \ldots + (p_{N-1} + p_{2N-1} + \ldots)e^{j\frac{2\pi m(N-1)}{N}}$$

$$= \sum_{k=0}^{N-1} p'_k e^{j\frac{2\pi km}{N}} \qquad \textit{with } p'_k = p_k + p_{N+k} + p_{2N+k} + \ldots$$

From inverse DFT we get $p'_0, p'_1, \ldots, p'_{N-1}$ which are equal to the desired p_k plus an error term e_k.

$$p'_0 = p_0 + e \quad \text{and} \quad e_k = p_{N+k} + p_{2N+k} + p_{3N+k} + \ldots$$

If N is large e_k can be made very small.

Ex: 5.35 X: geometric R.V. Find N such that percent error is 1%.

Recall that: $p_k = (1-p)p^k$

$$e_k = \sum_{h=1}^{\infty} p_{k+hN} = \sum_{h=1}^{\infty}(1-p)p^{k+hN} = (1-p)p^k \frac{p^N}{1-p^N}$$

$$\%\ \text{error} = \frac{e_k}{p_k} = \frac{p^N}{1-p^N} = a \cdot 100\%$$

$$\frac{p^N}{1-p^N} \le 0.01 \;\Rightarrow\; 100\,p^N \le 1 - p^N \quad \Rightarrow \quad 101 p^N \le 1$$

$$\Rightarrow \quad p^N \le \frac{1}{101} \quad \Rightarrow \quad N \log p \le -2$$

$$N > \frac{-2}{\log p}$$

>>>Sign change because of $p<1$ and $\log p<0$

Example:

p	N
.1	2
.5	7
.9	44

Continuous R.V.:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_X(w)\,e^{-jwx}\,dw \approx \frac{1}{2\pi}\sum_{m=-M}^{M-1}\Phi_X(mw_0)\,e^{-j\frac{2\pi nm}{N}} \qquad \text{for } -M \le n \le M-1$$

and

$$c_m = \frac{w_0}{2\pi}\Phi_X(mw_0)$$

See Ex: 5.36 for N = 512 p. 315

#5.1 U = X + Y + Z X,Y,Z zero-mean, $\sigma^2 = 1$
COV(X,Y) = 1/4 COV(Y,Z) = -1/4 COV(X,Z) = 0

a) Find mean & variance

E[U] = E[X + Y + Z] = E[X] + E[Y] + E[Z] = 0

$\sigma_U^2 = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 + 2COV(X,Y) + 2COV(X,Z) + 2COV(Y,Z)$
$= 1 + 1 + 1 + 2(1/4) + 2(0) + 2(-1/4) = 3$

b) X,Y,Z are uncorrelated

E[U] = 0
$\sigma_U^2 = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 + 0 + 0 + 0 = 3$

#5.3 $X_1, \ldots, X_n$ are R.V. with identical μ and $Cov(X_i, Y_j) = \sigma^2.\rho^{|i-j|}$. If $|\rho| < 1$ find $E[S_n]$ and $\sigma_{S_n}^2$

$$E[S_n] = n.\mu$$

Covariance Matrix is a Toeplitz matrix.

$$K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \rho^2\sigma^2 & \cdots & \rho^{n-1}\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \cdots & \rho^{n-2}\sigma^2 \\ \vdots & & & & \\ & & & & \\ \rho^{n-1}\sigma^2 & & \cdots & & \sigma^2 \end{bmatrix}$$

and

$$\sigma_{S_n}{}^2 = n\sigma^2 + 2\rho\sigma^2\sum_{j=1}^{n-1}\sum_{k=0}^{j-1}\rho^k = n\sigma^2 + 2\rho\sigma^2\sum_{j=1}^{n-1}\frac{1-\rho^j}{1-\rho}$$

$$= n\sigma^2 + 2\rho\sigma^2\left[\frac{n-1}{1-\rho} - \left(\frac{\rho}{1-\rho}\right)\frac{1-\rho^{n-1}}{1-\rho}\right]$$