Chapter 4: Multiple Random Variables

A vector r.v. **X** is a function that assigns a vector of real numbers to each outcome ξ in S : sample space.

Ex: 4.1 Student's features: ξ: name of a student in class and height, weight and age functions be:

$$\left.\begin{aligned} H(\xi) &= Height \\ W(\xi) &= Weight \\ A(\xi) &= Age \end{aligned}\right\}$$ then the vector: $(H(\xi), W(\xi), A(\xi))$ is a multi-random variable.

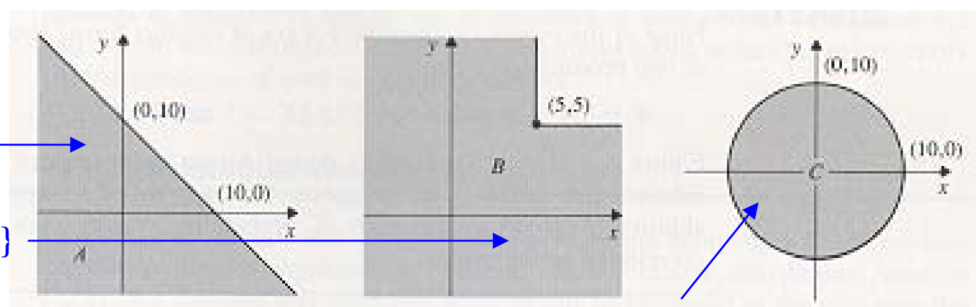
Ex: 4.3 ξ is outcome of a voltage waveform X(t)
$X_t = X(kT)$ be sample of voltage taken at t = kT
Then n samples form: $\mathbf{X} = (X_1, X_2, \ldots, X_n)$

Events involving an n-dimensional r.v. $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ has a corresponding region in an n-dim. space.

Ex: 4.4 Given **X** = (X, Y) Find the region of the plane corresponding to:

A = {X+Y≤10}

B = {min(X,Y) ≤5}



$\{\min(X,Y) \le 5\} = \{X \le 5\} \cup \{Y \le 5\}$ $C = \{X^2+Y^2 \le 100\}$

Product Form:

$A = \{X_1 \text{ in } A_1\} \cap \{X_2 \text{ in } A_2\} \cap .. \cap \{X_n \text{ in } A_n\}$. For r.v. $\mathbf{X} = \{X_1, X_2, .., X_n)$

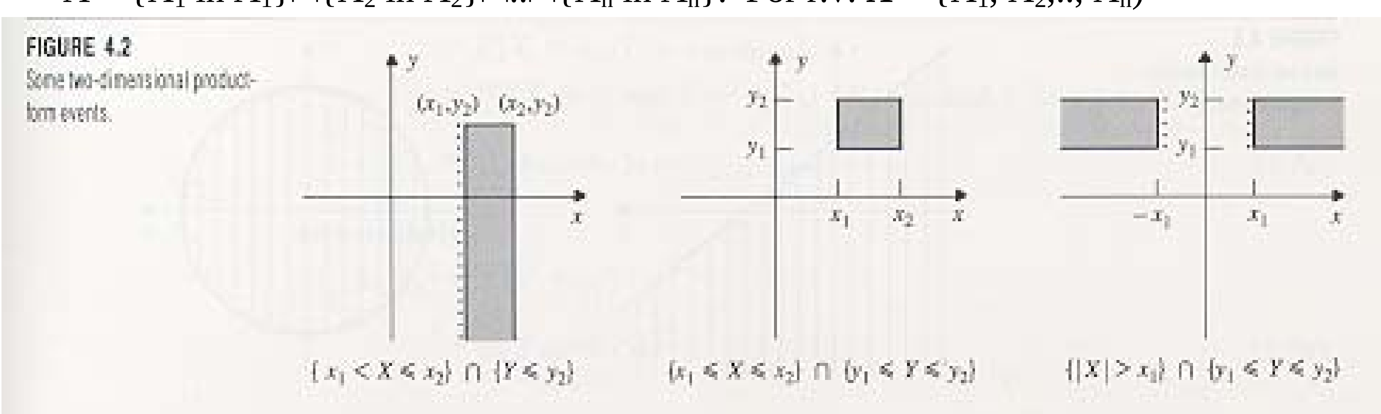
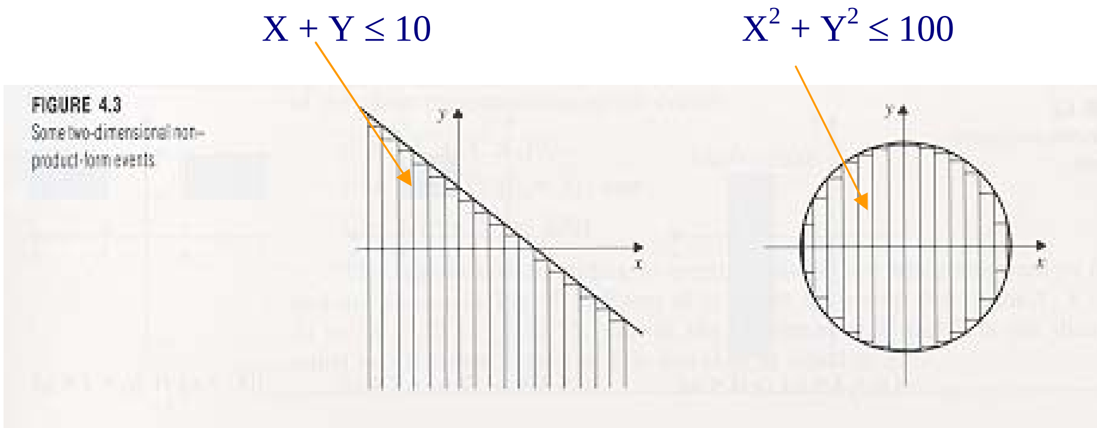


FIGURE 4.2 Some two-dimensional product-form events.

Probability of product-form events:

$P[A] = P[\{X_1 \text{ in } A_1\} \cap \{X_2 \text{ in } A_2\} \cap \ldots \cap \{X_n \text{ in } A_n\}]$ $\Rightarrow$ n-dim joint cdf, pdf
$\equiv P[X_1 \text{ in } A_1, X_2 \text{ in } A_2, \ldots, X_n \text{ in } A_n] = P[\{\xi \text{ in } S \text{ such that } X(\xi) \text{ in } A\}]$

Consider A, B, C of Fig 4.1 (Ex: 4.4) : None of these are of product form.
But B can be broken down to: $B = \{X \leq 5 \text{ and } Y < \infty\} \cup \{X > 5 \text{ and } Y \leq 5\}$
Approximating A & C by infinitesimal width rectangles:



R.V. $X_1, X_2, \ldots, X_n$ are **independent** if
$P[X_1 \text{ in } A_1, X_2 \text{ in } A_2, \ldots, X_n \text{ in } A_n] = P[X_1 \text{ in } A_1] \cdot P[X_2 \text{ in } A_2] \cdots P[X_n \text{ in } A_n]$

where A_k is an event that involves only X_k.

Two Random Variables:

Let $Z = (X, Y)$ take values from $S = \{(x_j, y_k)\,;\ j = 1, \ldots;\ k = 1, 2, \ldots\}$.

The **joint pmf** of Z specifies the probabilities of the product-form event:
$\{X = x_j\} \cap \{Y = y_k\}$

$p_{X,Y}(x_j, y_k) = P(\{X = x_j\} \cap \{Y = y_k\}) \equiv P[X = x_j, Y = y_k]$ $j = 1, \ldots; k = 1, \ldots$

1. Prob. of any event A is the sum of pmf over outcomes in A:

$$P[X \text{ in } A] = \sum_{(x_j, y_k)} \sum P(x_j, y_k)$$

2. Probability of S :

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{X,Y}(x_i, y_k) = 1$$

3. **Marginal pmf** (probabililty of events for each r.v.):

$$p_X(x_i) = \Pr ob[X = x_i] = \Pr ob[X = x_i;\ y = anything$$

$$p_X(x_i) = \Pr ob[\{X = x_i \quad and \quad y = y_1\} \cup \{X = x_i \quad and \quad y = y_2\} \cup \cdots]$$

$$p_X(x_i) = \sum_{k=1}^{\infty} p_{X,Y}(x_i, y_k)$$

and similarly, the marginal for y:

$$p_Y(y_k) = \Pr ob[Y = y_k] = \sum_{i=1}^{\infty} p_{X,Y}(x_i, y_k)$$

Ex: 4.6

A random experiment consists of tossing two "loaded" dice and noting the pair of numbers (X, Y) facing up. The joint pmf $p_{X,Y}(j, k)$ for $j = 1, \ldots, 6$ and $k = 1, \ldots, 6$ is:

j \ k	1	2	3	4	5	6
1	2/42	1/42	1/42	1/42	1/42	1/42
2	1/42	2/42	1/42	1/42	1/42	1/42
3	1/42	1/42	2/42	1/42	1/42	1/42
4	1/42	1/42	1/42	2/42	1/42	1/42
5	1/42	1/42	1/42	1/42	2/42	1/42
6	1/42	1/42	1/42	1/42	1/42	2/42

$$P[X = 1] = \frac{2}{42} + \frac{1}{42} + \ldots + \frac{1}{42} = \frac{1}{6}$$

Similarly,

$$P[X = j] = \frac{1}{6} \quad j = 1,2,3,4,5,6$$

and

$$P[Y = k] = \frac{1}{6} \quad k = 1,2,3,4,5,6$$

No way to tell from marginals the dice are loaded.

Ex: 4.7 If message length:*N-bytes* with a geometric distribution with probability *1-p* and $S_N = \{0,1,2,\ldots\}$. Pack them into M-bytes with Q of them, R-bytes left over. Find joint pmf, marginal pmf for Q, R.

$$P[Q = q, R = r] = P[N = qM + r] = (1 - p)p^{qM+r}$$

$$P[Q=q] = P[N \ in \ \{qM, qM+1, qM+2, ..., qM+(M-1)\}]$$

$$= \sum_{k=0}^{M-1} (1-p)p^{qM+k} = (1-p)p^{qM} \sum_{k=0}^{M-1} p^k = (1-p)p^{qM} \frac{1-p^M}{1-p}$$

$$= (1-p^M)(p^M)^q \quad for \ q = 0,1,2,..$$

$$P[R=r] = P[N \ in \ \{r, M+r, 2M+r, ...\}]$$

$$= \sum_{q=0}^{\infty} (1-p)p^{qM+r} = (1-p)p^r \sum_{q=0}^{\infty} (p^M)^q$$

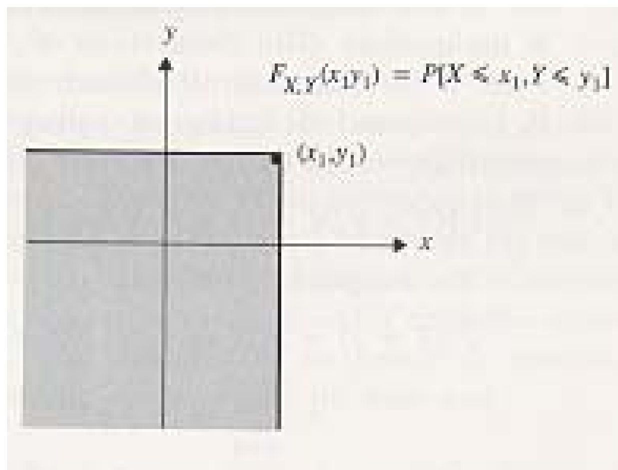
$$= (1-p)p^r \frac{1}{1-p^M} \quad for \ r = 0,1,2,...M-1$$

Probability of the product-form event:

$\{X \le x_1\} \cap \{Y \le y_1\}$

$F_{XY}(x_1,y_1) = P[X \le x_1, Y \le y_1]$

Joint cdf is non-decreasing in the "northeast" direction:

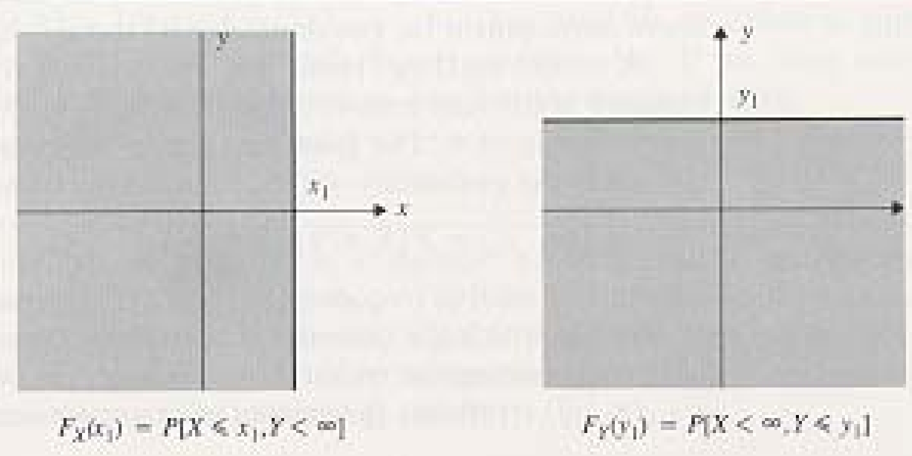


Joint cdf :

(i) $F_{XY}(x_1,y_1) \le F_{XY}(x_2,y_2)$ if $x_1 \le x_2$ and $y_1 \le y_2$

(ii) $F_{XY}(-\infty,y_1) = F_{XY}(x_1,-\infty) = 0$

FIGURE 4.5
The marginal cdf's are the probabilities of these half-planes.



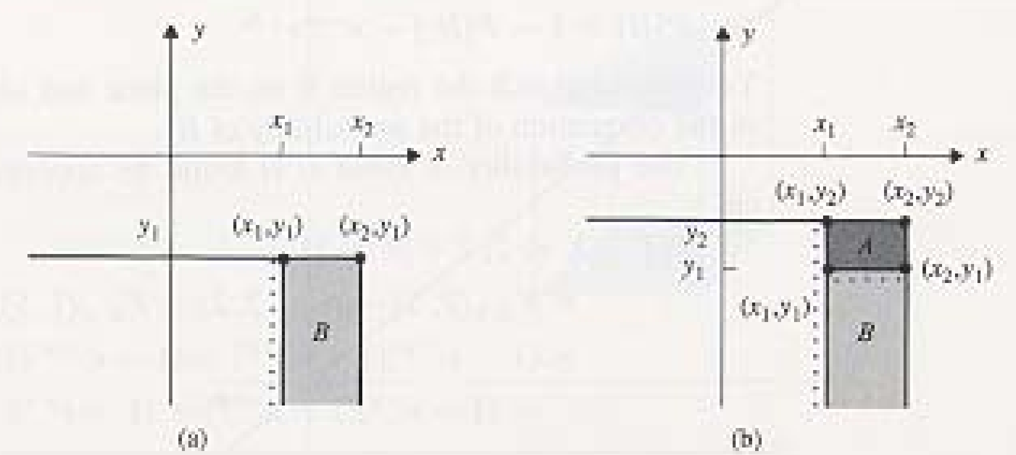
(iii) $F_{XY}(\infty,\infty) = 1$

(iv) Marginal cdf:

$F_x(x_1) = F_{XY}(x_1, \infty) = P[X \leq x_1, y < \infty)]$

$F_Y(y_1) = F_{XY}(\infty, y_1) = P[Y \leq y_1]$

FIGURE 4.6 The joint cdf can be used to determine the probability of various events.



(v) Continuous from the "north" and the "east"

(vi) $\lim_{x \to a^+} F_{XY}(x, y) = F_{XY}(a, y)$

and

$\lim_{y \to b^+} F_{XY}(x, y) = F_{XY}(x, b)$

(vii) $P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1)$
$- F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)$

Ex: 4.8 & 4.9:

$$F_{XY}(x, y) = \begin{cases} (1 - e^{-\alpha.x})(1 - e^{-\beta.y}) & x \geq 0; y \geq 0 \\ 0 & Otherwise \end{cases}$$

Find marginal cdf's

$$F_X(x) = \lim_{y \to \infty} F_{XY}(x, y) = (1 - e^{-\alpha.x})(1 - 0) = (1 - e^{-\alpha.x})$$

$$F_Y(y) = \lim_{x \to \infty} F_{XY}(x, y) = (1 - 0)(1 - e^{-\beta.y}) = (1 - e^{-\beta.y})$$

Find P(A), P(B), and P(D) if $A = \{X \leq 1, Y \leq 1\}$

$$P(A) = P[X \leq 1, Y \leq 1] = F_{XY}(1,1) = (1 - e^{-\alpha})(1 - e^{-\beta})$$

$B = \{X > x, Y > y\}$

From DeMorgan's Rule we write:

$B^C = (\{X > x\} \cap \{Y > y\})^C = \{X > x\}^C \cup \{Y > y\}^C$
$= \{X \le x\} \cup \{Y \le y\}$

$$P[B^C] = P[X \le x] + P[Y \le y] - P[X \le x, Y \le y]$$
$$= 1 - e^{-\alpha.x} + 1 - e^{-\beta.y} - (1 - e^{-\alpha.x})(1 - e^{-\beta.y})$$
$$= 1 - e^{-\alpha.x} e^{-\beta.y}$$

and $P[B] = 1 - P[B^C] = e^{-\alpha.x} e^{-\beta.y}$

$D = \{1 < X \le 2, 2 < Y \le 5\}$

$$P(D) = F_{XY}(2,5) - F_{XY}(2,2) - F_{XY}(1,5) + F_{XY}(1,2)$$
$$= (1 - e^{-2\alpha})(1 - e^{-5\beta}) - (1 - e^{-2\alpha})(1 - e^{-2\beta}) - (1 - e^{-\alpha})(1 - e^{-5\beta}) + (1 - e^{-\alpha})(1 - e^{-2\beta})$$

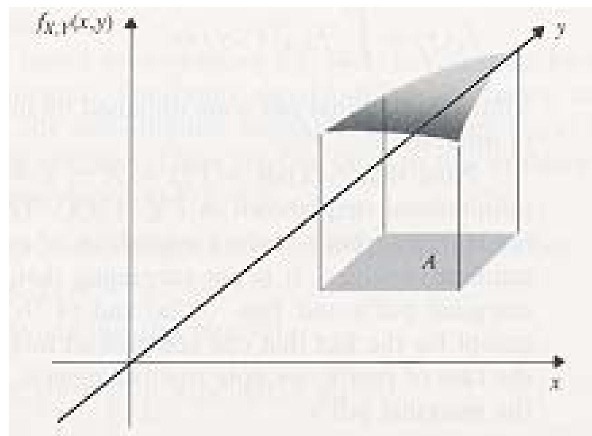
Joint pdf of two continuous r.v.:

X and **Y** are jointly continuous if there is a joint pdf $f_{XY}(x,y) \ge 0$ and it is defined in the real plane such that for every event A

$$P[X \text{ in } A] = \iint_A f_{XY}(x,y)dxdy$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)dxdy$$

$$f_{XY}(x,y) = \frac{d^2}{dxdy} F_{X,Y}(x,y)$$



If we define $A = \{(x,y): a_1 < X \le b_1 \text{ and } a_2 < Y \le b_2\}$ then,

$$P[A] = P[a_1 < X \le b_1, a_2 < Y \le b_2] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{XY}(x,y)dxdy$$

and

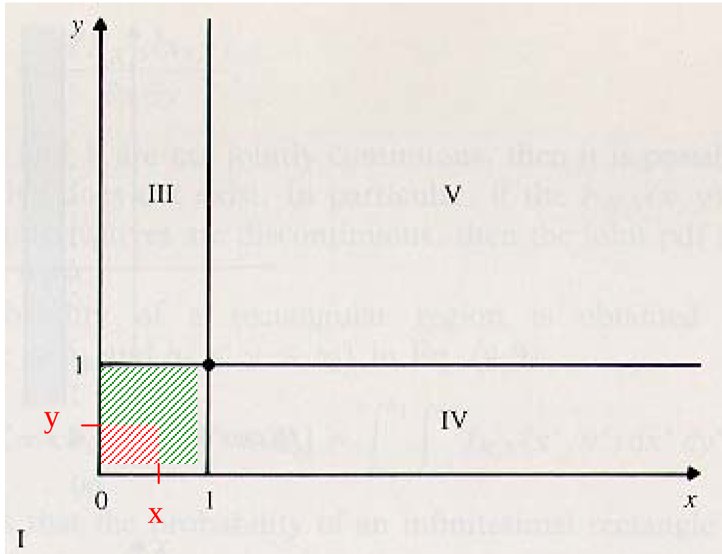
$$P[x < X \le x + dx,\ y < Y \le y + dy] \approx f_{XY}(x,y)dxdy$$

which leads to:

1. $F_X(x) = F_{XY}(x,\infty)$ and $F_Y(y) = F_{XY}(\infty, y)$
2. $f_X(x) = \dfrac{d}{dx} \displaystyle\int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f_{XY}(x,y)dy \right] dx = \left[\int_{-\infty}^{\infty} f_{XY}(x,y)dy \right]$
3. $f_y(y) = \dfrac{d}{dy} \displaystyle\int_{-\infty}^{y} \left[\int_{-\infty}^{\infty} f_{XY}(x,y)dx \right] dy = \left[\int_{-\infty}^{\infty} f_{XY}(x,y)dx \right]$

Ex: 4.10 Given:

$$f(x,y)=\begin{cases}1 & \begin{cases}0\le x\le 1\\ 0\le y\le 1\end{cases}\\ 0 & o.w.\end{cases}$$



Find joint cdf

Region I: Since x < 0 , y < 0

$f_{XY} \rightarrow 0$ then: $F_{XY}(x,y) = 0$

Region II: Where $0 \le x \le 1$ and $0 \le y \le 1$:

$$F_{XY} = \int_0^x \int_0^y 1 \cdot dxdy = xy$$

Region III: Where $0 \le x \le 1$ and y > 1:

$$F_{XY} = \int_0^x \int_0^1 dxdy = x$$

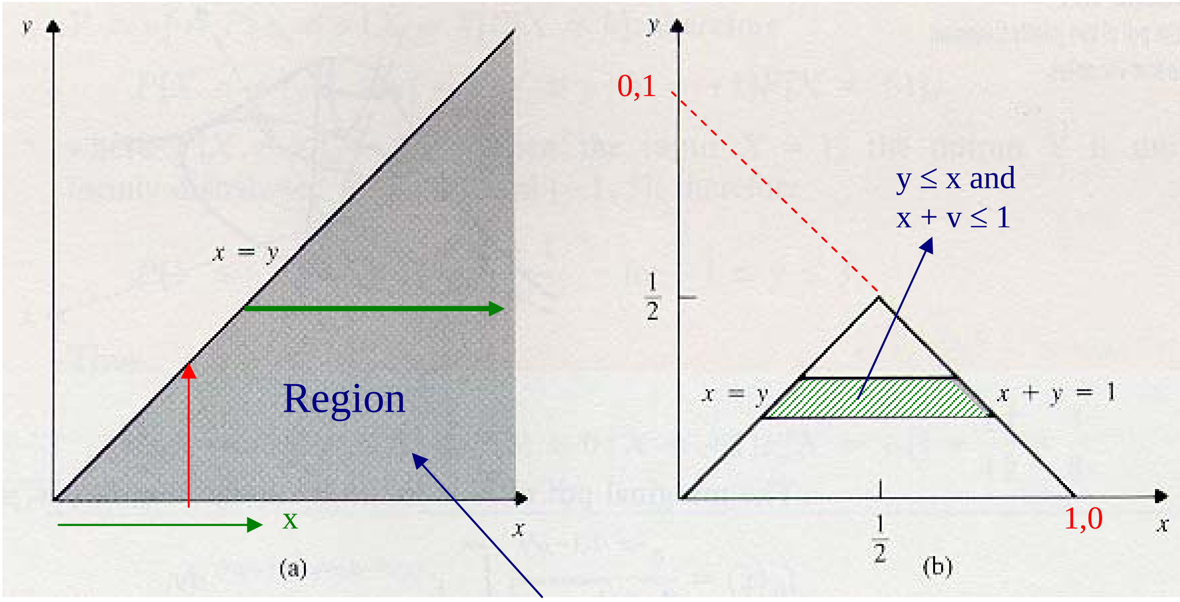
Region IV: Where x > 1 but $0 \le y \le 1$:

$$F_{XY} = \int_0^y \int_0^1 dxdy = y$$

Region V: Where x > 1 and y > 1:

$$F_{XY} = \int_0^1 \int_0^1 dxdy = 1$$

Ex: 4.11 and 4.12:



Find **c**:

$$1 = \int_0^\infty dx \int_0^x c\ e^{-x} e^{-y} dy = \int_0^\infty c\ e^{-x}(1-e^{-x})dx$$

$$= c\int_0^\infty e^{-x}dx - c\int_0^\infty e^{-2x}dx = c - \frac{c}{2} = \frac{c}{2} \quad \Rightarrow \quad c = 2$$

Marginal pdfs:

$$f_X(x) = \int_0^\infty f_{XY}(x,y)dy = 2\int_0^x e^{-x} e^{-y}dy = 2\,e^{-x}(1-e^{-x}) \quad 0 \le x < \infty$$

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)dx = 2\int_y^\infty e^{-x} e^{-y}dx = 2\,e^{-2y} \quad 0 \le y < \infty$$

$$f_{XY}(x,y) = \begin{cases} ce^{-x} e^{-y} & 0 \le y \le x \le \infty \\ 0 & Otherwise \end{cases}$$

If $x + y \le 1$ find $P[x+y \le 1]$ ⇐ **The region is the strip**

$$P[x+y \le 1] = \int_0^{1/2} \left(\int_y^{1-y} 2\ e^{-x} e^{-y} dx \right) dy = \int_0^{1/2} 2e^{-y}\left[e^{-y} - e^{-(1-y)}\right]dy$$

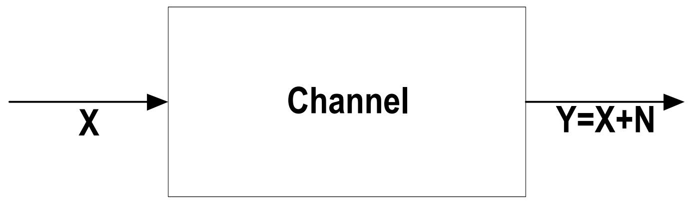
$$= 2\int_0^{1/2} e^{-2y}dy - 2\int_0^{1/2} e^{-1}dy = 1 - 2e^{-1}$$

Mixed r.v.

$\Rightarrow$ $P[X = k; Y \leq y]$; where y: cont. r.v. but x: discrete r.v.

$P[X = k; y_1 < Y \leq y_2]$

Ex: 4.14



Input: X={1,-1} with Prob:1/2

and

Noise: N: Uniform in [-2V, +2V]

Find $P[X = 1, Y \leq 0]$ Case for Sent "1" but received "0", i.e., ERROR

$$P[X = k, Y \leq y] = P[Y \leq y \mid X = k]P[X = k]$$

When X = 1 then Y is uniform in (-2+1, 2+1) = (-1, 3)

then

$$P[Y \leq y \mid X = 1] = 1/4 \quad \text{and} \quad P[X = 1] = 1/2$$

$$P[X = 1, Y \leq 0] = P[Y \leq 0 \mid X = 1]P[X = 1]$$
$$= (1/4)(1/2) = 1/8$$

Independent 2 R.V.:

$$P[X_1 \text{ in } A_1, X_2 \text{ in } A_2] = P[X_1 \text{ in } A_1]\, P[X_2 \text{ in } A_2]$$

For discrete r.v.:

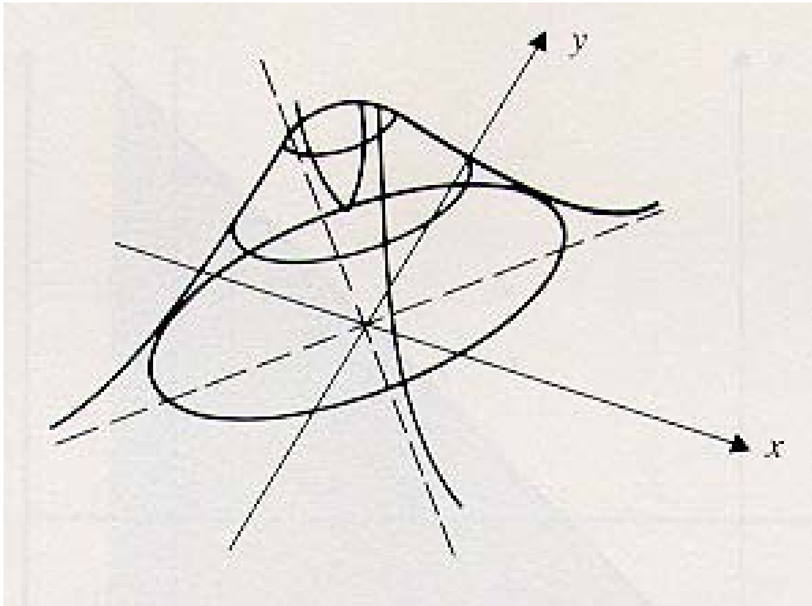
$$p_{XY}(x_j, y_k) = P[X = x_j, Y = y_k] = P[X = x_j]\, P[Y = y_k]$$
$$= p_X(x_j)\, p_Y(y_k) \quad \text{for all } x_j, y_k$$

$\Rightarrow$ If X and Y are indep discrete r.v. then their joint pmf is the product of marginal pmfs.

$\Rightarrow$ If X and Y are jointly continuous and if $f_{XY}(x,y) = f_X(x)\, f_Y(y)$ for all x and y; then x and y are independent.

$\Rightarrow$ In general, X and Y are indep r.v. iff: $F_{XY}(x,y) = F_X(x)\, F_Y(y)$ for all x, y

Ex: 4.13 & 4.18 Given X,Y two jointly Gaussian r.v. where $-\infty < X,Y < \infty$, find marginal pdf's and check for independence.



$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}$$

$$f_X(x) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} e^{-\frac{y^2 - 2\rho xy}{2(1-\rho^2)}} dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2 - \rho^2 x^2}{2(1-\rho^2)}} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}}{\sqrt{2\pi(1-\rho^2)}} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$f_X(x) \cdot f_Y(y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} = f_{XY}(x,y) \qquad \text{iff} \quad \Rightarrow \quad \rho = 0$$

$$\text{where} \quad \rho \equiv \text{correlation coefficient}$$

Ex: 4.15 Does the "loaded die" experiment (Ex: 4.6) have indep. r.v.?

Recall: P[X = Y = j] = 2/42 and P[X = j, Y = k; k ≠ j] = 1/42

But

P[X = j]P[Y = k] = (1/6)(1/6) = 1/36 for all j,k

Therefore, X & Y are not independent.

Ex: 4.16 Are Q and R in Ex.7 independent?

Using

$$P[Q=q]=(1-p^M)(p^M)^q \qquad q=0,1,2,\ldots$$

$$P[R=r]=\frac{1-p}{1-p^M}p^r \qquad r=0,1,\ldots,M-1$$

$$P[Q=q]P[R=r]=(1-p^M)(p^M)^q\frac{1-p}{1-p^M}p^r=(1-p)^{Mq+r}$$

$$=P[Q=q,R=r] \qquad \text{for all } q=0,1,2,\ldots \quad r=0,1,\ldots,M-1$$

Therefore, Q and R independent.

⇒ If X and Y are indep r.v., then the r.v. defined by any pair of functions g(X) and h(Y) are also indep.

⇒ P[g(X) in A, h(Y) in B] = P[X in A′, Y in B′]
= P[X in A′]P[Y in B′]
= P[g(X) in A]P[h(Y) in B]

Conditional Prob. & Cond. Expectations:

Prob. that Y is in A given that X = x is known:

$$P[Y \text{ in } A \mid X=x]=\frac{P[Y \text{ in } A, X=x]}{P[X=x]} \qquad P[X=x]\neq 0$$

If X is discrete then cond. cdf of Y given X = x_k:

$$F_Y(y\mid x_k)=\frac{P[Y\le y, X=x_k]}{P[X=x_k]}$$

and the cond. pdf of Y given given X = x_k , if derivative exists

$$f_Y(y\mid x_k)=\frac{\partial}{\partial y}F_Y(y\mid x_k)$$

Prob. of an event A given X = x_k is $P[Y \text{ in } A \mid X=x_k]=\int_{y \text{ in } A} f_Y(y\mid x_k)dy$

If X an Y are discrete then cond. pmf of Y given X = x_k :

$$p_Y(y_j \mid x_k) = P[Y = y_j \mid X = x_k] = \frac{P[X = x_k, Y = y_j]}{P[X = x_k]}$$

$$= \frac{p_{XY}(x_k, y_j)}{p_x(x_k)} \qquad if\ \ P[X = x_k] > 0$$

If X and Y are independent:

$$P_Y(y_j \mid x_k) = \frac{P[X = x_k, Y = y_j]}{P[X = x_k]} = P[Y = y_j] = P_Y(y_j)$$

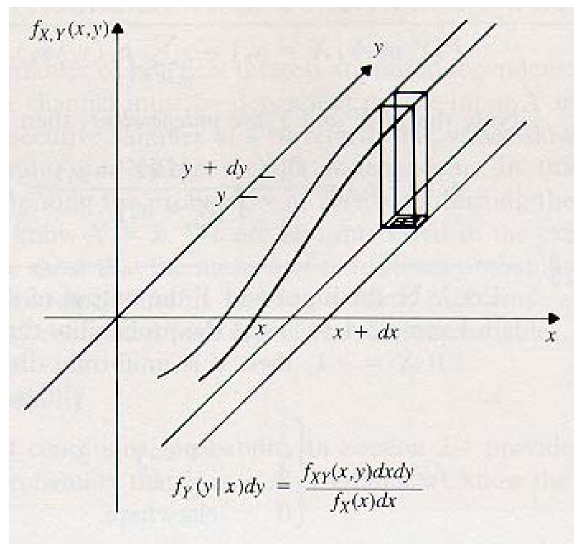
If X and Y are continuous then conditional cdf of Y given X = x :

$$F_Y(y \mid x) = \lim_{h \to 0} F_Y(y \mid x < X \le x + h)$$

but

$$F_Y(y \mid x < X \le x + h) = \frac{P[Y \le y, x < X \le x + h]}{P[x < X \le x + h]}$$

$$= \frac{\int_{-\infty}^{y} \int_{x}^{x+h} f_{XY}(x', y')dx'dy'}{\int_{x}^{x+h} f_X(x')dx'} = \frac{\int_{-\infty}^{y} f_{XY}(x, y}{f_X(x}$$



let $h \to 0$ *then*

$$F_Y(y \mid x) = \frac{\int_{-\infty}^{y} f_{XY}(x, y')dy'}{f_X(x)}$$

and conditional pdf of Y given X = x

$$f_Y(y \mid x) = \frac{d}{dy} F_Y(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

If X and Y are independent, then we have: $f_{XY}(x, y) = f_X(x)\, f_Y(y)$ and

$$f_Y(y \mid x) = f_Y(y) \quad and \quad F_Y(y \mid x) = F_Y(y)$$

Conditional Expectation of Y given X=x

$$E[Y \mid x] = \int_{-\infty}^{\infty} y\, f_Y(y \mid x)dy \qquad if\ \ continuous$$

$$E[Y \mid x] = \sum_{y_i} y_i \; p_Y(y_i \mid x) \qquad \textit{if } X \;\&\; Y \textit{ are both discrete}$$

Notes:

1. If X is continuous: $E[Y] = E\{E[Y \mid X]\} = \int_{-\infty}^{\infty} E[Y \mid X] \, f_X(x) dx$
2. If X is discrete: $E[Y] = E\{E[Y \mid X]\} = \sum_{x_k} E[Y \mid x_k] \, p_X(x_k)$
3. Furthermore: E[h(y)] = E{E[h(y) | X]} and $E[y^k] = E\{E[y^k \mid X]\}$.

Multiple Random Variables:

Joint cdf of $X_1, X_2, \ldots, X_n$:

$$F_{X_1 \ldots X_n}(x_1, x_2, \ldots, x_n) \equiv P[X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n]$$

It is defined for continuous, discrete, or mixed type r.v.

1. Joint pmf of n-dim. Discrete r.v. :

$$p_{X_1 \ldots X_n}(x_1, x_2, \ldots, x_n) = P[X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n]$$

2. Marginal pmf:

$$p_{X_j}(x_j) = P\lfloor X_j = x_j \rfloor = \sum_{x_1} \cdot \; \cdot \; \cdot \sum_{x_n} p_{X_1 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

$$\textit{but } x_j \textit{ is excluded}$$

3. Conditional pmfs:

$$p_{X_n}(x_n \mid x_1, x_2, \ldots, x_{n-1}) = \frac{p_{X_1 \ldots X_n}(x_1, x_2, \ldots, x_n)}{p_{X_1 \ldots X_{n-1}}(x_1, x_2, \ldots, x_{n-1})} \qquad \textit{if } \; p_{X_1 \ldots X_{n-1}}(x_1, x_2, \ldots, x_{n-1}) > 0$$

Repeated usage of the above yields:

$$p_{X_1 \ldots X_n}(x_1, x_2, \ldots, x_n) = p_{X_n}(x_n \mid x_1, x_2, \ldots, x_{n-1}) \, p_{X_{n-1}}(x_{n-1} \mid x_1, x_2, \ldots, x_{n-2}) \cdots$$
$$\cdots p_{X_2}(x_2 \mid x_1) \, p_{X_1}(x_1)$$

4. If $X_1, \ldots, X_n$ are jointly continuous r.v., then Joint pdf and cdf:

$$F_{X_1 \ldots X_n}(x_1, \ldots x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1 \ldots X_n}(x_1', \ldots x_n') dx_1' \cdots dx_n'$$

and

$$f_{X_1 \ldots X_n}(x_1, \ldots x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{X_1 \ldots X_n}(x_1, \ldots x_n)$$

5. Marginal pdfs are found by integration:

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 \ldots X_n}(x_1', \ldots x_n') dx_2' \cdots dx_n'$$

6. Conditional pdfs:

$$f_{X_1 \ldots X_n}(x_n \mid x_1, \ldots x_{n-1}) = \frac{f_{X_1 \ldots X_n}(x_1, \ldots x_n)}{f_{X_1 \ldots X_{n-1}}(x_1, \ldots x_{n-1})} \quad if \;\; f_{X_1 \ldots X_{n-1}}(x_1, \ldots x_{n-1}) > 0$$

and repeated usage results in:

$$f_{X_1 \ldots X_n}(x_1, \ldots x_n) = f_{X_1 \ldots X_n}(x_n \mid x_1, \ldots x_{n-1}) \cdots f_{X_2}(x_2 \mid x_1)\, f_{X_1}(x_1)$$

$$f_{X_1 \ldots X_n}(x_1, \ldots x_n) = f_{X_n}(x_n \mid x_1, \ldots x_{n-1}) \cdots f_{X_2}(x_2 \mid x_1)\, f_{X_1}(x_1)$$

Independence: $X_1, \ldots, X_n$ are independent if

$P[X_1 \text{ in } A_1, \ldots, X_n \text{ in } A_n] = P[X_1 \text{ in } A_1]P[X_1 \text{ in } A_1] \cdots P[X_n \text{ in } A_n]$

Equivalently, if

$$F_{X_1 \ldots X_n}(x_1, \ldots x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

1. If all X_i are discrete then if independent

$$p_{X_1 \ldots X_n}(x_1, \ldots x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$$

2. If all X_i are continuous then if independent:

$$f_{X_1 \ldots X_n}(x_1, \ldots x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

Ex: 4.29:

For jointly Gaussian pdf:

$$f_{X_1 X_2 X_3}(x_1, x_2, x_3) = \frac{1}{2\pi\sqrt{\pi}} e^{-\left(x_1^2 + x_2^2 - \sqrt{2} x_1 x_2 + \frac{1}{2} x_3^2 \right)}.$$

Find marginal pdf of X_1 and X_3.

$$f_{X_1 X_3}(x_1, x_3) = \frac{e^{-\frac{x_3^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\frac{2\pi}{\sqrt{2}}} e^{-\left(x_1^2 + x_2^2 - \sqrt{2} x_1 x_2 \right)} dx_2$$

Using Ex: 4.13 result with $\rho = \frac{1}{\sqrt{2}}$ the integral yields:

$$f_{X_1 X_3}(x_1, x_3) = \frac{e^{-\frac{x_3^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{x_1^2}{2}}}{\sqrt{2\pi}}$$

⇒ X_1 and X_3 are independent Gaussian random variables with N(0,1) [zero mean, unit variance]

Ex: 4.30: White Noise Gaussian signal samples with pdf:

$$f_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)}$$

forms a product of n one-dimensional Gaussian pdfs with independent identical mean and variance N(0,1).

Functions of Several Random Variables

Single function of Multiple R.V.: $Z = g(X_1, X_2, \ldots, X_n)$

1. **cdf of Z:** Let us find the equivalent event of $\{Z \le z\}$

⇒ $R_z = \{\mathbf{x}: (x_1, \ldots, x_n)$ such that $g(\mathbf{x}) \le z\}$

$$F_Z(z) = P[X \text{ in } R_z] = \int_{\mathbf{x} \text{ in } R_Z} \cdots \int f_{X_1 \ldots X_n}(x_1', \ldots, x_n') dx_1' \cdots dx_n'$$

2. **pdf of Z:** $f_Z(z) = \frac{d}{dz} F_Z(z)$

3. **Conditional pdf:** pdf of Z given Y = y:

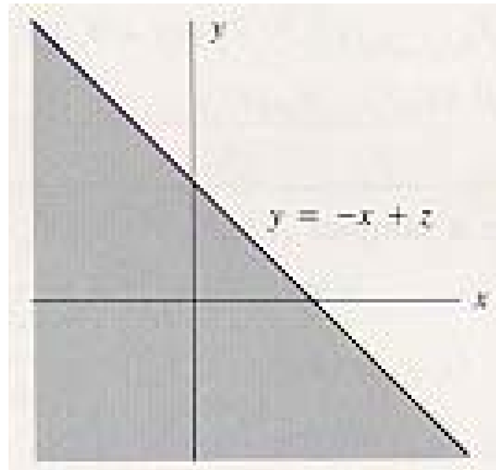
$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z \mid y') f_Y(y') dy'$$

Ex: 4.31,32: Z = X + Y. Find $F_Z(z)$ and $f_Z(z)$

$$F_Z(z) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{-x'+z} f_{XY}(x', y')dy$$

and pdf :

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x', z - x')dx'$$



If X and Y are independent, we obtain:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z - x)dx$$ **⇒Convolution integral**

If X and Y be Gaussian N(0,1) with $\rho = -1/2$, then

$$f_{XY}(x, y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}$$

and

$$f_Z(z) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\rho x(z-x) + (z-x)^2}{2(1-\rho^2)}} dx$$

after manipulation we re-write:

$$f_Z(z) = \frac{1}{2\pi(3/4)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - xz + z^2}{2(3/4)}} dx \Rightarrow \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

⇒ **Sum of two Gaussian R.V. (not necessarily independent) is also Gaussian**

Ex: 4.34 Given Z = X/Y ; where X and Y are independent r.v. with exponential distribution and E[X] & E[Y] = 1. Find $f_z(z)$.

Let Y = y , then Z = X/y is a scaled version of X and from Ex: 3.23:

$$f_Z(z/y) = |y|\, f_X(yz \mid y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_X(yz \mid y)\, f_Y(y)dy = \int_{-\infty}^{\infty} |y|\, f_{XY}(yz, y)dy$$

$$f_Z(z) = \int_0^\infty y\, f_X(yz)\, f_Y(y) dy \qquad \textit{for } z > 0$$

$$= \int_0^\infty y\, e^{-yz} e^{-y} dy = \int_0^\infty y\, e^{-y(1+z)} dy = \frac{1}{(1+z)^2}$$

N-functions of N-variables (Transformation):

$$Z_1 = g_1(X), \dots, Z_n = g_n(X)$$

then

$$F_{Z_1 \dots Z_n}(z_1, \dots, z_n) = P\left[g_1(X) \le z_1, \dots, g_n(X) \le z_n\right]$$

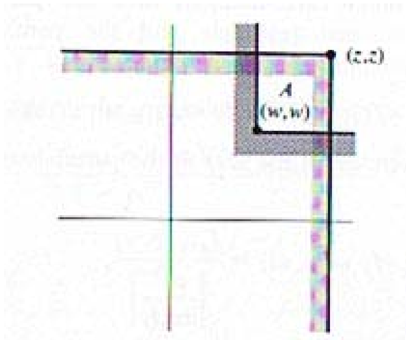
Assume continuous r.v. then

$$F_{Z_1 \dots Z_n}(z_1, \dots, z_n) = \int_{\mathbf{X}' : g_k(\mathbf{X}') \le z_k} \cdots \int f_{X_1 \dots X_n}(x_1', \dots, x_n') dx_1' \cdots dx_n'$$

Ex: 4.35 If we have two new variables defined as $W = min(X,Y)$ and $Z = max(X,Y)$ then let us find $F_{ZW}(z,w)$.

Consider Figure 4-14 in the text for

$$\{\min(X,Y) \le w\} = \{X \le w\} \cup \{Y \le w\} \text{ and } \{\max(X,Y) \le z\} = \{X \le z\} \cup \{Y \le z\}$$



$$F_{ZW}(z,w) = P[\{\min(X,Y) \le w\} \cap \{\max(X,Y) \le z\}]$$

1. If $z > w$, then

$$P_{ZW}(z,w) = F_{XY}(z,z) - P[A]$$

$$= F_{XY}(z,z) - \{ F_{XY}(z,z) - F_{XY}(w,z) - F_{XY}(z,w) + F_{XY}(w,w)\}$$

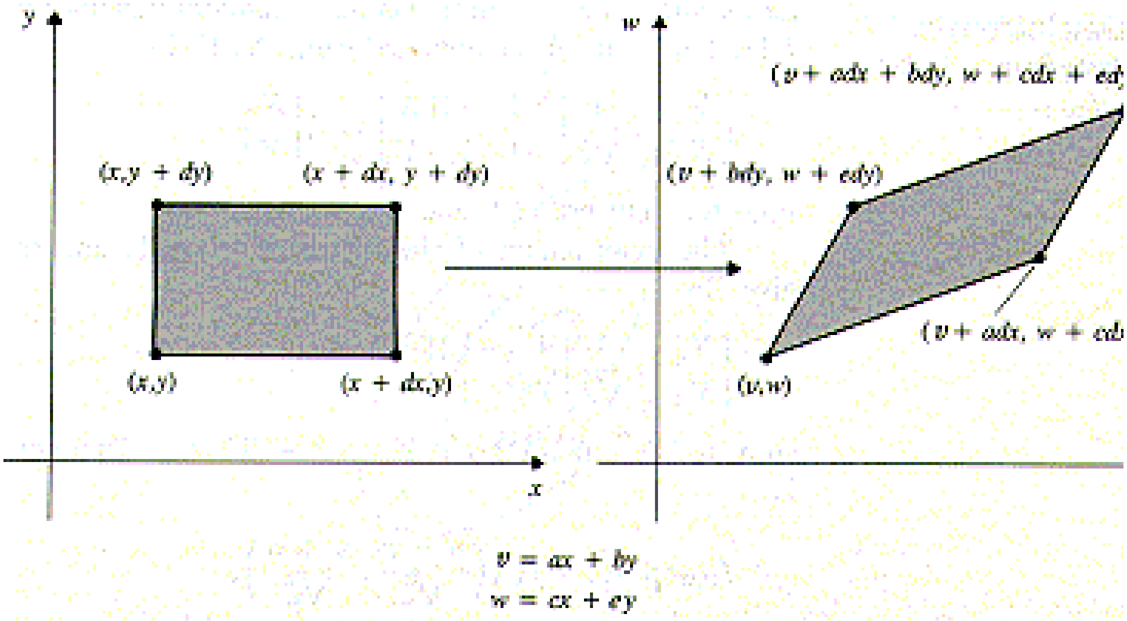
2. If $z<w$, then:

$$F_{WZ}(w,z) = F_{XY}(z,z)$$

Linear Transformations: Consider a pair of equations:

V = aX + bY   and   W = cX + eY

Let us re-write them in matrix form: $\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = [A]\begin{bmatrix} X \\ Y \end{bmatrix}$



Let us assume that the inverse A^{-1} exists then: $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} v \\ w \end{bmatrix}$ will be the representation in *(v,w)* space. Consider infinitesimal rectangle, where

dP = area of parallelogram

then

$$f_{XY}(x,y)dxdy \approx f_{VW}(v,w)dP$$

$$\Rightarrow f_{VW}(v,w) = f_{XY}(x,y)\left|\frac{dxdy}{dP}\right| \qquad \text{"stretch factor"}$$

$$dP = |ae - bc|(dxdy) \qquad \Rightarrow \left|\frac{dP}{dxdy}\right| = \frac{|ae-bc|(dxdy)}{(dxdy)} = |ae - bc| = |A|$$

which results in:

$$f_{VW}(v,w) = \frac{f_{XY}(x,y)}{|A|}\Bigg|_{\left\{ x \atop y \right\} = A^{-1}\left\{ v \atop w \right\}}$$

For n-dimensional R.V. linear transformations: Z = AX

$$f_Z(z) \equiv f_{Z_1 \dots Z_n}(z_1, \dots, z_n) = \left. \frac{f_{X_1 \dots X_n}(x_1, \dots, x_n)}{|A|} \right|_{\underset{\sim}{x} = A^{-1} \underset{\sim}{z}} = \frac{f_{\underset{\sim}{X}}(A^{-1} \underset{\sim}{z})}{|A|}$$

Ex: 4.36: Given X & Y are jointly Gaussian with:

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} \quad \text{and}$$

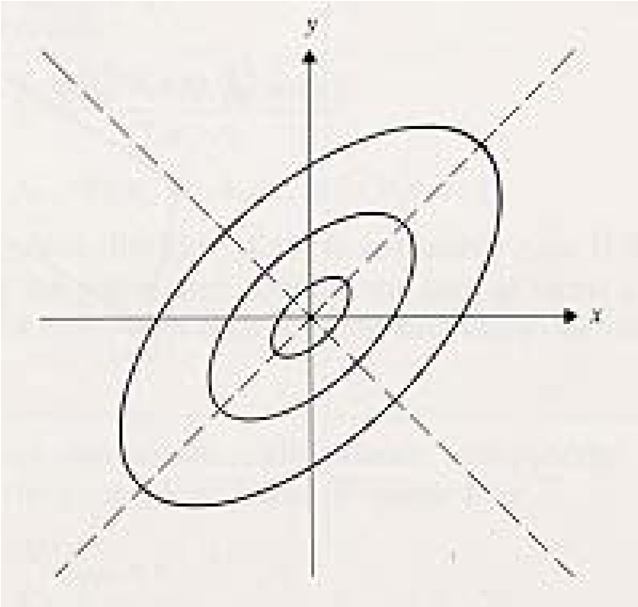
$$\begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{with} \quad A^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} \Rightarrow X = \frac{V - W}{\sqrt{2}} \quad Y = \frac{V + W}{\sqrt{2}}$$

Therefore:

$$f_{VW}(v,w) = f_{XY}\left(\frac{v-w}{\sqrt{2}}, \frac{v+w}{\sqrt{2}} \right) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left\{ \frac{v^2}{2(1-\rho)} + \frac{w^2}{2(1-\rho)} \right\}}$$

Rotating coordinates to coincide with that of the ellipsoid



General Transformations:

V = g$_1$(X,Y)
W = g$_2$(X,Y)

Assume: v(x,y) and w(x,y) are invertible with:

x = h$_1$(v,w) and **y = h$_2$(v.w)**

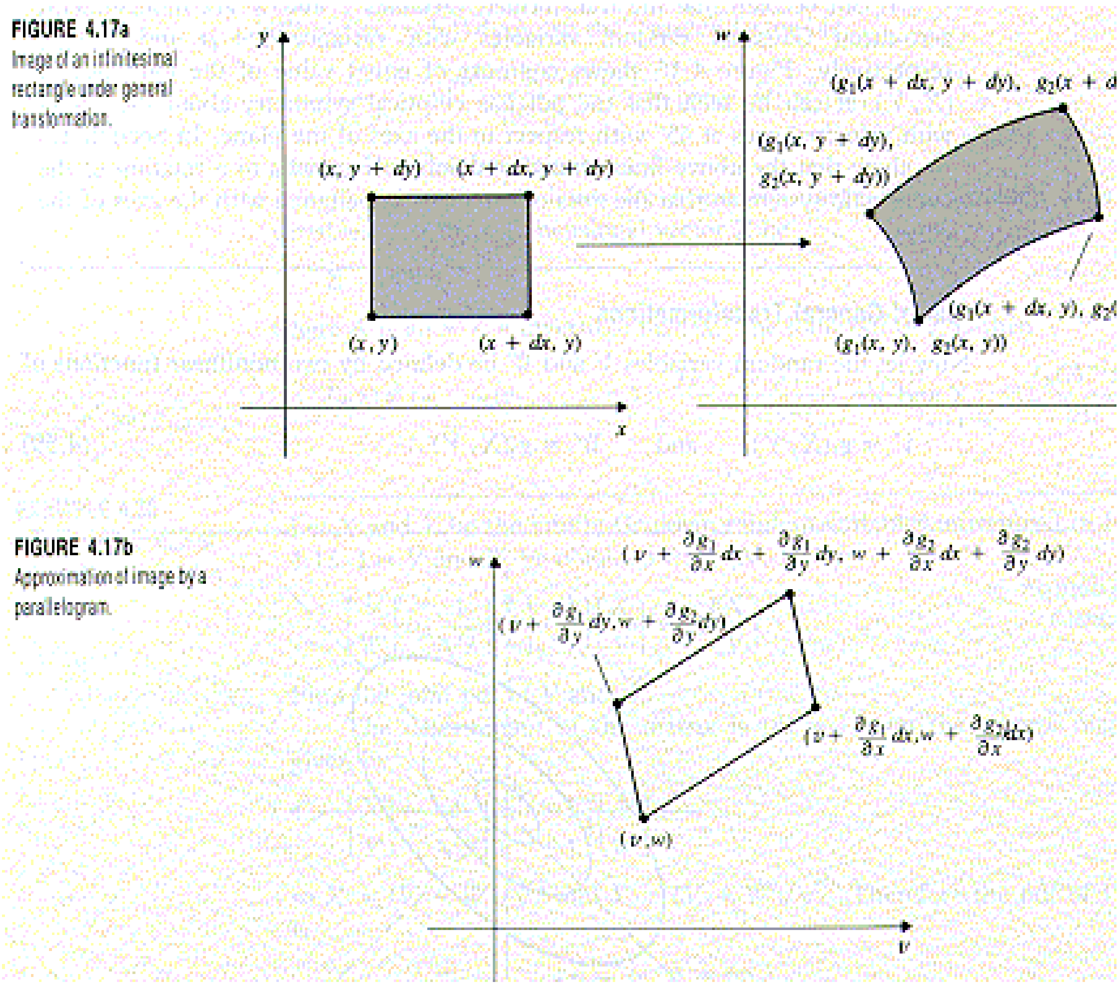


FIGURE 4.17a Image of an infinitesimal rectangle under general transformation.

FIGURE 4.17b Approximation of image by a parallelogram.

Then the image of a small rectangle in (x,y) can be approximated by a parallelogram in (v,w) and

$$f_{XY}(x,y)dxdy = f_{VW}(v,w)dP \Rightarrow f_{VW}(v,w) = \frac{f_{XY}(h_1,h_2)}{\left|\frac{dP}{dxdy}\right|}$$

⇒ "Stretch factor" represented by Jacobian:

$$J(x,y) = \det\begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \quad and \quad J(v,w) = \det\begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{bmatrix}$$

with property:

$$|J(v,w)| = \frac{1}{|J(x,y)|}$$

Finally:

$$f_{VW}(v,w) = \frac{f_{XY}(h_1(v,w), h_2(v,w))}{|J(x,y)|} = f_{XY}(h_1(v,w), h_2(v,w))|J(v,w)|$$

and for multi-variable transformations we have:

$$\mathbf{X} = (X_1,\ldots,X_n) \quad \Rightarrow \quad \mathbf{Z} \; with \; Z_1 = g_1(\mathbf{X}),..,Z_n = g_n(\mathbf{X})$$

$$f_{Z_1\ldots Z_n}(z_1,\ldots,z_n) = \frac{f_{X_1\ldots X_n}(h_1,\ldots,h_n)}{|J(x_1,\ldots,x_n)|} = f_{X_1\ldots X_n}(h_1,\ldots,h_n)|J(z_1,\ldots z_n)|$$

with:

$$|J(x_1,\ldots,x_n)| = \det\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & . & . & . & \frac{\partial g_1}{\partial x_n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial g_n}{\partial x_1} & . & . & . & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

Ex: 4.37: Given: X & Y

Assume both variables are zero-mean, with unity variance, $\sigma^2 = 1$, independent Gaussian R.V. Find pdf of V and W where:

$$V = \sqrt{X^2 + Y^2} \quad \text{and} \quad W = angle(X,Y) \; in \; (0,2\pi)$$

Cartesian to polar mapping:

$$x = v\cos w \quad \text{and} \quad y = v\sin w \quad \Rightarrow \quad \begin{cases} v = \sqrt{x^2 + y^2} \\ w = \arctan\left(\frac{y}{x}\right) \end{cases}$$

$$J(v,w)=\det\begin{bmatrix}\frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\end{bmatrix}=\det\begin{bmatrix}\cos w & -v\sin w\\ \sin w & v\cos w\end{bmatrix}=v\cos^2 w+v\sin^2 w=v$$

Therefore,

$$f_{VW}(v,w)=\frac{v}{\sqrt{2\pi}}e^{-\frac{v^2\cos^2 w}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2\sin^2 w}{2}}=\frac{v}{2\pi}e^{-\frac{v^2}{2}\left(\cos^2 w+\sin^2 w\right)}$$

$$=\frac{v}{2\pi}e^{-\frac{v^2}{2}}\quad \text{for } v\geq 0 \quad \text{and} \quad 0\leq w<2\pi$$

$\Rightarrow$ **Rayleigh pdf radius, V and uniform in W: (0, 2π)**

Expected values of Z = g(X,Y)

1. $E[Z]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)dxdy$ where (x,y) both continuous.
2. $E[Z]=\sum_i\sum_n g(x_i,y_n)\,P_{XY}(x_i,y_n)$ where (x,y) both discrete.
3. $E[X_1+X_2+...+X_n]=E[X_1]+E[X_2]+...+E[X_n]$
4. If $g(x,y) = g_1(x)\cdot g_2(y)$ and X and Y are independent, then

   $E[g_1(x)\cdot g_2(y)] = E[g(x,y)] = E[g_1(x)]\cdot E[g_2(y)]$
5. (j,k)th joint moment of X and Y:

   $E\left[X^jY^k\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{XY}(x,y)dxdy$ if (x,y) both continuous

   $E\left[X^jY^k\right]=\sum_i\sum_n x_i^j y_n^k\, P_{XY}(x_i,y_n)$ if (x,y) both discrete

Special Cases:

1) $j = 0$ $E[X^0\, Y^k] = E[Y^k]$
2) $k = 0$ $E[X^j\, Y^0] = E[X^j]$
3) $j = k = 1$ E[XY] = correlation of X and Y
4) $j = k = 1$ $E[XY] = E[X]\cdot E[Y]$ then X and Y are uncorrelated
5) $j = k = 1$ E[XY] = 0 then X and Y are orthogonal
6) (j,k)th central moment of X and Y: $E\left[(X-E[X])^j(Y-E[Y])^k\right]$
7) $j = 2$, $k = 0$: $E\left[(X-E[X])^2\right]=\sigma_X^2$

8) j = 0, k = 2 : $E\left[(Y - E[Y])^2\right] = \sigma_Y^2$

9) j = 1, k = 1 :

$$E\left[(X - E[X])(Y - E[Y])\right] = Cov(X,Y) = E[XY] - E[X]E[Y]$$

10) If E[X] = 0 or E[Y] = 0 , then cov(X,Y) = E[XY]

Correlation Coefficient ≡ $\rho_{XY} = \dfrac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \dfrac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$

11) Note that: $-1 \le \rho_{XY} \le 1$

12) If X and Y are linearly related: Y = aX + b , then

$$\rho_{XY} = \begin{cases} 1 & if\ a > 0 \\ -1 & if\ a < 0 \end{cases}$$

13) If $\rho_{XY} = 0$ then, X and Y are uncorrelated

14) If X & Y are independent, then $\rho_{XY} = \text{cov}(X,Y) = 0$

Lemma 1: If X and Y are independent they are ALWAYS uncorrelated. (Converse is not necessarily true)

Lemma 2: If X and Y are jointly Gaussian then independence and uncorrelatedness imply each other.

Ex: 4.39: For Z = X + Y find E[Z]

$$E[Z] = E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x'+y')\, f_{XY}(x',y')dx'dy'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x'\, f_{XY}(x',y')dx'dy' + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y'\, f_{XY}(x',y')dx'dy'$$

$$= \int_{-\infty}^{\infty}x'\, f_X(x')dx' + \int_{-\infty}^{\infty}y'\, f_Y(y')dy' = E[X] + E[Y]$$

Ex: 4.41 Let X & Y be independent, then

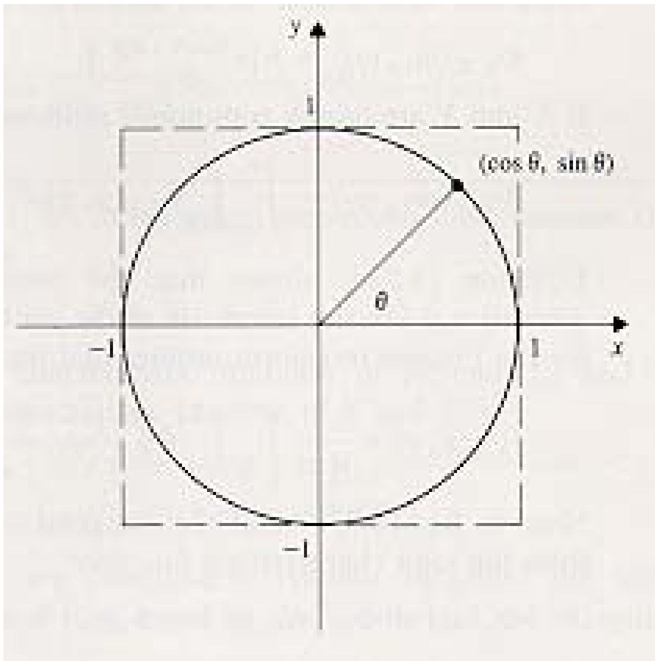
$$\text{cov}(X,Y) = E\left[(X - E\{X\})(Y - E\{Y\})\right] = E\left[X - E\{X\}\right]E\left[Y - E\{Y\}\right] = 0$$

Similarly,

$$\text{cov}(X,Y) = 0 \quad \Rightarrow \quad \text{cov}(X,Y) = E[XY - E[X]E[Y] = 0$$

$$\Rightarrow \quad E[XY] = E[X]E[Y]$$

Ex: 4.42 Given uniform phase with $p_\theta(\theta) = \begin{cases} 1/2\pi & 0 \le \theta < 2\pi \\ 0 & o.w. \end{cases}$ and $X = Cos\theta$ and $Y = Sin\theta$; polar coordinates.



$$E[XY] = E[Sin\theta.Cos\theta] = \int_0^{2\pi} \frac{1}{2\pi} Sin\theta.Cos\theta.d\theta = \frac{1}{4\pi}\int_0^{2\pi} Sin2\theta.d\theta = 0$$

$\Rightarrow$ **X and Y are uncorrelated.**

Joint Characteristic Function:

$$\Phi_{X_1 \ldots X_n}(w_1, \ldots, w_n) = E\left[e^{j(w_1 x_1 + \ldots + w_n x_n)}\right]$$

If X and Y are both continuous then

$$\Phi_{XY}(w_1, w_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) e^{j(w_1 x + w_2 y)} dxdy$$

and Inverse Fourier Transform yields:

$$f_{XY}(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{XY}(w_1, w_2) e^{-j(w_1 x + w_2 y)} dw_1 dw_2$$

with moments:

$$E\left[X^i Y^k\right] = \frac{1}{j^{(i+k)}} \frac{\partial^i \partial^k}{\partial w_1^{\,i} \partial w_2^{\,k}} \Phi_{XY}(w_1, w_2)\Big|_{\substack{w_1=0 \\ w_2=0}}$$

1) $\Phi_X(w) = \Phi_{XY}(w,0)$ and $\Phi_Y(w) = \Phi_{XY}(0,w)$

2) If X and Y are independent, then

$$\Phi_{XY}(w_1, w_2) = \Phi_X(w_1)\Phi_Y(w_2)$$

3) If Z = aX + bY, then

$$\Phi_Z(w) = E\left[e^{jw(aX+bY)}\right] = \Phi_{XY}(aw, bw)$$

4) If X and Y are independent and Z = aX + bY, then

$$\Phi_Z(w) = \Phi_{XY}(aw, bw) = \Phi_X(aw)\cdot\Phi_Y(bw)$$

Jointly Gaussian Normal R.V.

$$f_{XY}(x,y) = \frac{\exp\left\{\frac{1}{2\left(1-\rho^2{}_{XY}\right)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho_{XY}\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2{}_{XY}}}$$

(See Figure 4.19 Page: 239)

n-Jointly Gaussian R.V.

$$f_{\mathbf{X}}(\mathbf{x}) \equiv f_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}|\mathbf{K}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{K}^{-1}(\mathbf{x}-\mathbf{m})\right\}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \quad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ . \\ . \\ . \\ m_n \end{bmatrix} \quad and \quad \mathbf{K} = \begin{bmatrix} VAR(x_1) & COV(x_1 x_2) & . & . & . & COV(x_1 x_n) \\ COV(x_2 x_1) & VAR(x_2) & . & . & . & COV(x_2 x_n) \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ COV(x_n x_1) & & & & & VAR(x_n) \end{bmatrix}$$

Ex: 4.48 **X** is jointly Gaussian with COV($X_j X_i$) = 0 if i ≠ j, then we show that **X** = X_1, X_2, …, X_n are independent r.v.

$$\mathbf{K} = \begin{bmatrix} VAR(x_1) & 0 & . & . & . & 0 \\ 0 & VAR(x_2) & . & . & . & 0 \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ 0 & & & & & VAR(x_n) \end{bmatrix} = diag[VAR(x_i)] = diag[\sigma_i{}^2]$$

$$\Rightarrow \quad \mathbf{K^{-1}} = diag\left[\frac{1}{\sigma_i{}^2}\right] \quad and \quad (\mathbf{x}-\mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x}-\mathbf{m}) = \sum_{i=1}^{n}\left(\frac{x_i - m_i}{\sigma_i}\right)^2$$

and

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\mathbf{K}|^{1/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - m_i}{\sigma_i}\right)^2\right\} = \prod_{i=1}^{n}\frac{e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2}}{\sqrt{2\pi\sigma_i{}^2}}$$

$$\therefore \quad f_{\mathbf{X}}(\mathbf{x}) = \prod f_{X_i}(x_i) \quad \Rightarrow \quad X_1, X_2, \ldots X_n \quad \textit{are independent r.v.}$$

Linear Transformation of Gaussian R.V.:

Let **X** = $X_1, X_2, \ldots, X_n$ be jointly Gaussian and **Y** = **AX** where **A** has an inverse.

Then

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{|\mathbf{A}|} f_{\mathbf{X}}(\mathbf{A}^{-1}\mathbf{y}) = \frac{1}{(2\pi)^{n/2}|\mathbf{K}|^{1/2}|\mathbf{A}|} \exp\left\{-\frac{1}{2}[\mathbf{A^{-1}y}-\mathbf{m}]^T \mathbf{K}^{-1}[\mathbf{A^{-1}y}-\mathbf{m}]\right\}$$

After using linear algebra identities, we have

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}|\mathbf{C}|^{1/2}} \exp\left\{-\frac{1}{2}[\mathbf{y}-\mathbf{n}]^T \mathbf{C}^{-1}[\mathbf{y}-\mathbf{n}]\right\}$$

where

$$\mathbf{n} = \mathbf{Am} \quad \mathbf{C} = \mathbf{AKA}^T \quad and \quad \det(\mathbf{C}) = \det(\mathbf{A})^2 \det(\mathbf{K})$$

Finally,

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\sqrt{(2\pi\lambda_1)(2\pi\lambda_2)\cdots(2\pi\lambda_n)}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i - n_i}{\lambda_i}\right)^2\right\}$$

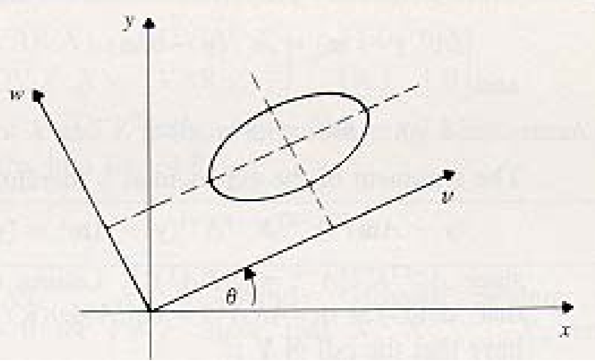
where $\lambda_1,...,\lambda_n$ are diagonal components of $\mathbf{\Lambda} = \mathbf{AKA}^T$.

It is possible to linearly transform a vector of jointly Gaussian r.v. into a **vector of independent Gaussian r.v.**

Ex: 4.49

FIGURE 4.20
A rotation of the coordinate system transforms a pair of dependent Gaussian random variables into a pair of independent Gaussian random variables.

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}\right)$$



Let New Coordinate System be: $\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$

X and Y are independent if COV(X,Y) = 0.

$COV(V,W) = E[(V - E\{V\})(W - E\{W\})$

$$= E[\{(X - m_1)\cos\theta + (Y - m_2)\sin\theta\}\{-(X - m_1)\sin\theta + (Y - m_2)\cos\theta\}]$$

$$= -\sigma_1^2 \sin\theta\cos\theta + COV(X,Y)\cos^2\theta - COV(X,Y)\sin^2\theta + \sigma_2^2 \sin\theta\cos\theta$$

$$= \frac{1}{2}\{Sin2\theta[(\sigma_2^2 - \sigma_1^2) + 2Cos2\theta.COV(X,Y)\}$$

$$= \frac{Cos2\theta}{Cos2\theta}\{\} = Cos2\theta.\{\tan 2\theta.(\sigma_2^2 - \sigma_1^2) + 2Cov(X,Y)\}.\frac{1}{2}$$

Let θ be such that: $\tan 2\theta = \dfrac{2COV(X,Y)}{\sigma_1^2 - \sigma_2^2}$

$$COV(V,W) = \frac{1}{2}\left\{\cos 2\theta\left[(\sigma_2^2 - \sigma_1^2)\frac{2COV(X,Y)}{\sigma_1^2 - \sigma_2^2} + 2COV(X,Y)\right]\right\}$$

COV(V,W) = 0 **$\therefore$ X and Y are independent.**

Mean Square Estimation

- Estimating parameters of one or more r.v. and estimating the value of an inaccessible r.v. Y in terms of the observation of an accessible r.v. X.
- Estimating Y from a function of X, where the estimation error, ε = Y – g(X) , is non-zero and a cost function C(Y-g(X)) is associated with the process.

The usual form of C is mean square error is given by:

$$C = E[\{Y - g(X)\}^2]$$

Task: Find $C_{\min}$

1) Estimate a r.v. Y by a constant α such that C is minimum:

$$\min_{\alpha} E\{(Y-\alpha)^2\} = \min_{\alpha}\{E[Y^2] - 2\alpha E[Y] + \alpha^2\}$$

$$\frac{dC}{d\alpha} = 2\alpha - 2E[Y] \quad \to 0 \quad \Rightarrow \alpha^* = E[Y]$$

$$C_{\min} = E[(Y-\alpha^*)^2] = E[(Y - E\{Y\})^2] = \sigma_Y{}^2$$

2) Estimate Y by g(X) = αX + β

$$C_{\min} = \min_{\alpha,\beta} E\left[(Y - \alpha X - \beta)^2\right] = \min_{\alpha,\beta} E\left[(\{Y - \alpha X\} - \beta)^2\right]$$

which is same as minimization of Y- αX by a constant β:

$$\beta^* = E[Y - \alpha X] = E[Y] - \alpha E[X]$$

$$C_{\min} = \min_{\alpha} E\left[\{(Y - E[Y]) - \alpha(X - E[X])\}^2\right]$$

$$\frac{dC}{d\alpha} = \frac{d}{d\alpha} E\left[(Y - E[Y]) - \alpha(X - E[X])\right]^2 = 0$$

$$= -2E\left[\{(Y - E[Y]) - \alpha(X - E[X])\}(X - E[X])\right]$$

$$-2COV(X,Y) + \alpha VAR(X) \quad \Rightarrow 0$$

$$\alpha^* = \frac{COV(X,Y)}{VAR(X)} = \rho_{XY}\frac{\sigma_Y}{\sigma_X}$$

Therefore:

$$\hat{Y} = \alpha^* X + \beta^* = \rho_{XY} \frac{\sigma_Y}{\sigma_X} (X - E[X]) + E[Y]$$

and the mse of the best linear estimator (from orthogonality condition):

$$C_{\min} = VAR(Y)\left(1 - \rho^2{}_{XY}\right)$$

3. Orthogonality Condition:

$$E\left[\left\{(Y - E[Y]) - \alpha^* (X - E[X])\right\}(X - E[X])\right] = 0$$

The error of the best linear estimator is **orthogonal** to observation X-E[X].

4. ρ_{XY} specifies the sign and extent of the estimate of Y relative to:

$$\sigma_Y (X - E[X]) / \sigma_X$$

5. If X and Y are uncorrelated then,

$$\rho_{XY} = 0 \quad \Rightarrow \quad \hat{Y} = E[Y]$$

6. If $\rho_{XY} = \pm 1$, then

$$\hat{Y} = \pm \frac{\sigma_Y}{\sigma_X} \left(X - E[X]\right) + E[Y]$$

7. **Nonlinear Estimator:**

$$C_{\min} = \min_{g(\bullet)} E\left[(Y - g(X))^2\right]$$

It is obtained from regression analysis and generally have smaller mse in comparison with linear estimators but they are inherently more difficult to obtain.

#4.3 Let X, Y, Z be independent r.v. Find prob. in terms of F_X, F_Y, F_Z

a)

$$P[|X| < 5, Y > 2, Z^2 \geq 2] = P[|X| < 5]P[Y > 2]P[Z^2 \geq 2]$$

$$= P[-5 < X < 5](1 - P[Y \leq 2])(1 - P[-\sqrt{2} < Z < \sqrt{2}]$$

$$= [F_X(5^-) - F_X(-5)].(1 - F_Y(2)).[1 - F_Z(\sqrt{2^-}) + F_Z(-\sqrt{2})]$$

b)

$$P[X > 5, Y < 0, Z = 1] = P[X > 5]P[Y < 0]P[Z = 1]$$

$$= [1 - F_X(5^-)].[F_Y(0^-)].[F_Z(1) - F_Z(1^-)$$

c) $P[\min(X,Y,Z) > 2] = P[X > 2, Y > 2, Z > 2]$
$= P[X > 2]P[Y > 2]P[Z > 2]$
$= (1 - F_X(2))(1 - F_Y(2))(1 - F_Z(2))$

d) $P[\max(X,Y,Z) < 6] = P[X < 6, Y < 6, Z < 6]$
$= P[X < 6]P[Y < 6]P[Z < 6]$
$= [F_X(6^-)].[F_Y(6^-)].[F_Z(6^-)]$

#4.9 X & Y amplitude of noise signals at two antennas with

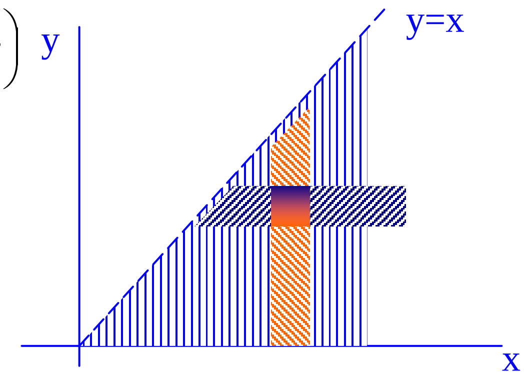
$$f_{XY}(x,y) = abxy\, e^{-ax^2/2} e^{-by^2/2} \qquad x > 0,\ y > 0,\ a > 0,\ b > 0$$

a) Find joint cdf

$$F_{XY}(x,y) = \int_0^x \int_0^y ax\, e^{-ax^2/2}\, by\, e^{-by^2/2}\, dydx$$

$$= \int_0^x axe^{-ax^2/2}\, dx \int_0^y by\, e^{-by^2/2}\, dy$$

$$= \left(1 - e^{-ax^2/2}\right)\left(1 - e^{-by^2/2}\right)$$



b) Find P[X>Y]

$$P[X > Y] = \int_0^\infty dx \int_0^x ax\, e^{-ax^2/2}\, by\, e^{-by^2/2}\, dy$$

$$= \int_0^\infty ax\, e^{-ax^2/2}\, dx \int_0^x by\, e^{-by^2/2}\, dy$$

$$= \int_0^\infty ax\, e^{-ax^2/2} \left(1 - e^{-bx^2/2}\right) dx$$

$$= \qquad = \int_0^\infty ax\, e^{-ax^2/2}\, dx - a \int_0^\infty xe^{-(b+a)x^2/2} dx$$

$$= 1 - \frac{a}{a+b}$$

Find marginal pdfs:

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = \lim_{y\to\infty}\left(1-e^{-ax^2/2}\right)\left(1-e^{-by^2/2}\right)$$

$$= 1-e^{-ax^2/2}$$

$$\Rightarrow f_X(x) = \frac{d}{dx}F_X(x) = ax\,e^{-ax^2/2}$$

Similarly,

$$f_Y(y) = by\,e^{-by^2/2}$$

#4.20 Are X & Y independent in #4.9?

Given: $f_{XY}(x,y) = abxy\,e^{-ax^2/2}e^{-by^2/2}$

X & Y are independent if $f_{XY}(x,y) = f_X(x)\,f_Y(y)$

But we found that $f_X(x) = ax\,e^{-ax^2/2}$ and $f_Y(y) = by\,e^{-by^2/2}$

Thus: $f_X(x)\,f_Y(y) = abxy\,e^{-ax^2/2}\cdot e^{-by^2/2} = f_{XY}(x,y)$ *Q.E.D.*

#4.25 X & Y are jointly Gaussian with $N(m_1,\sigma_1^2)$; $N(m_2,\sigma_2^2)$

a) Show X & Y are independent if $\rho = 0$.

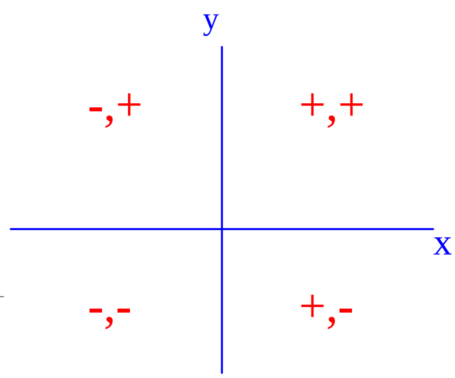
$$f_{XY}(x,y) = \frac{\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right)+\left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

If $\rho = 0$, then

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2}e^{\frac{-\left[\left(\frac{x-m_1}{\sigma_1}\right)^2+\left(\frac{y-m_2}{\sigma_2}\right)^2\right]}{2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1\sigma_1}}e^{\frac{-\left(\frac{x-m_1}{\sigma_1}\right)^2}{2}}\frac{1}{\sqrt{2\pi\sigma_2\sigma_2}}e^{\frac{-\left(\frac{y-m_2}{\sigma_2}\right)^2}{2}}$$

$$= f_X(x)\,f_Y(y) \quad \textit{for all } x, y$$



⇒ X,Y independent R.V.'s

Find P[XY > 0]

P[XY > 0] = P[X and Y have same sign]

$$= \iint(+,+) f_{XY}(x,y)dxdy + \iint(-,-) f_{XY}(x,y)dxdy$$

$$= \int_0^\infty f_X(x)dx \int_0^\infty f_Y(y)dy + \int_{-\infty}^0 f_X(x)dx \int_{-\infty}^0 f_Y(y)dy$$

using

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-m_1)^2}{2\sigma^2{}_1}} dx = \int_{-\frac{m_1}{\sigma_1}}^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt = Q\left(-\frac{m_1}{\sigma_1}\right)$$

and similarly for other integrals, we obtain:

$$P[XY > 0] = Q\left(-\frac{m_1}{\sigma_1}\right)Q\left(-\frac{m_2}{\sigma_2}\right) + \left(1 - Q\left(-\frac{m_1}{\sigma_1}\right)\right)\left(1 - Q\left(-\frac{m_2}{\sigma_2}\right)\right)$$

#4.35 P(C = i) = p_i and T = time to service a customer is exp. distr with α

a) Find pdf of T

$$f_T(t) = \sum_{i=1}^n f_T(t \mid C = i)P[C = i] = \sum_{i=1}^n \alpha_i e^{-\alpha_i t} p_i \qquad for\ t > 0$$

b) Find E[T] and $\sigma_T{}^2$

$$E[T] = \sum_{i=1}^n E[T \mid i]P[C = i] = \sum_{i=1}^n \frac{1}{\alpha_i} p_i$$

$$E[T^2] = \sum_{i=1}^n E[T^2 \mid i]P[C = i] = \sum_{i=1}^n \frac{2}{\alpha_i{}^2} p_i$$

$$VAR[T] = E[T^2] - E[T]^2 = \sum_{i=1}^n \frac{2}{\alpha_i{}^2} p_i - \left(\sum_{i=1}^n \frac{1}{\alpha_i} p_i\right)^2$$

#4.41 Show that

$$f_{XYZ}(x,y,z) = f_Z(z \mid x,y) f_Y(y \mid x) f_X(x)$$

From Bayes Theorem we have:

$$f_{XYZ}(x,y,z) = f_Z(z \mid x,y) f_{XY}(x,y) = f_Z(z \mid x,y) f_Y(y \mid x) f_X(x)$$

Q.E.D.

#4.42 Let U_1, U_2, U_3 be independent r.v. with
$X = U_1$; $Y = U_1 + U_2$; $Z = U_1 + U_2 + U_3$

a) Find joint pdf of X,Y,Z.

$$f_Y(y \mid x) = f_{U_2}(y - u_1) = f_{U_2}(y - x)$$

$$f_Z(z \mid x, y) = f_{U_3}(z - u_1 - u_2) = f_{U_3}(z - y)$$

Therefore,

$$f_{XYZ}(x, y, z) = f_Z(z \mid x, y)\, f_Y(y \mid x)\, f_X(x)$$

$$= f_{U_3}(z - y)\, f_{U_2}(y - x)\, f_{U_1}(x)$$

b) If U_i are independent uniform in [0,1],then find pdf of Y & Z; pdf of Z.

$$f_{YZ}(y, z) = \int_{-\infty}^{\infty} f_{U_3}(z - y)\, f_{U_2}(y - x)\, f_{U_1}(x)dx$$

$$= f_{U_3}(z - y) \underbrace{\int_{-\infty}^{\infty} f_{U_2}(y - x)\, f_{U_1}(x)dx}_{f_Y(y)}$$

from convolution Integral of (4.54) page 222.

$$= f_{U_3}(z - y)\, f_Y(y)$$

Find $f_Y(y)$ $\quad$ *for* $0 \le y \le 1$

$$f_Y(y) = \int_0^y f_{U_2}(y - u_1) \cdot f_{U_1}(u_1)du_1 = \int_0^y 1 \cdot 1 \cdot du_1 = y$$

for $1 \le y \le 2$

$$f_Y(y) = \int_{y-1}^{1} f_{U_2}(y - u_1) \cdot f_{U_1}(u_1)du_1 = 2 - y$$

$$\therefore \quad f_y(y) = \begin{cases} y & 0 \le y \le 1 \\ 2 - y & 1 \le y \le 2 \\ 0 & o.w. \end{cases}$$

and

$$f_{yz}(y,z)=\begin{cases} y & 0\le y\le 1;\ y\le z\le y+1 \\ 2-y & 1\le y\le 2;\ y\le z\le y+1 \\ 0 & o.w. \end{cases}$$

pdf Z:

$$f_Z(z)=\int f_{yz}(y',z)dy'=\begin{cases} \int_0^z y\,dy=\frac{1}{2}z^2 & 0\le z\le 1 \\ \int_{z-1}^{1} y\,dy+\int_1^z (2-y)dy=z^2-3z+\frac{3}{2} & 1\le z\le 2 \end{cases}$$