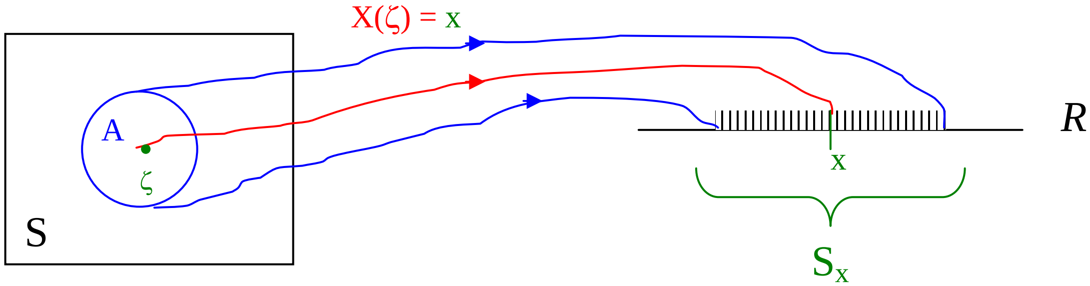
Chapter 3: Random Variables

Consider an Experiment with a Sample space S with outcomes ξ



- A random variable X is a function that maps each outcome of an experiment to a real number X(ζ).
- S is the domain of x and the set S_x is the ensemble of all values taken by X and called range of X

Ex: 3.3 Consider coin-tossing, where {X = k} = { k heads in 3 coin tosses}

S = { hhh, … , ttt} S_x = {0, 1, 2, 3}

$P_0 = P[x = 0] = (1-p)^3$ since $P\{TTT\} = (1-p)^3$

$P_1 = P[x = 1] = 3(1-p)^2 p$

$P_2 = P[x = 2] = 3(1-p)p^2$

$P_3 = P[x = 3] = p^3$

- If A is the set of outcomes ξ in S that lead to values X(ξ) in B : A = {ξ : X(ξ) in B} , then B in S_x occurs whenever A in S occurs. Then

$$P(B) = P(A) = P[\{\xi : X(\xi) \text{ in } B\}]$$

and A and B are equivalent events in different spaces.

Cumulative Distribution/Probability Density Functions

Cumulative Distribution Function (cdf) of *X* is defined by:

$$F_X(x) = P[X \le x] \quad \text{for } -\infty < x < \infty$$

Probability Density Function (pdf) is defined as:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Both cdf : $F_x(x)$ and pdf $f_x(x)$ are functions of the real variable *x*.

Axioms and Properties:

1. $0 \le F_x(x) \le 1$ and $f_x(x) \ge 0$

2. $\lim_{x\to\infty} F_X(\mathrm{x}) = 1$ and $\int_{-\infty}^{\infty} f_X(\mathrm{x})d\mathrm{x} = 1$

3. $\lim_{x\to-\infty} F_X(\mathrm{x}) = 0$ and $F_X(x) = \int_{-\infty}^{x} f(x')dx'$

4. $F_x(x)$ is a non-decreasing function of **x**, in other words

   $a < b$ then: $F_x(a) \le F_x(b)$ and $P[a \le X \le b] = P[X = a\} + P[a < X \le b] = \int_a^b f_X(x)dx$

5. $F_x(x)$ is continuous from the right, in other words for $h > 0$

   $$F_X(b) = \lim_{h\to 0} F_X(b+h) = F_X(b^+)$$

6. $P[a < X \le b] = F_x(b) - F_x(a)$

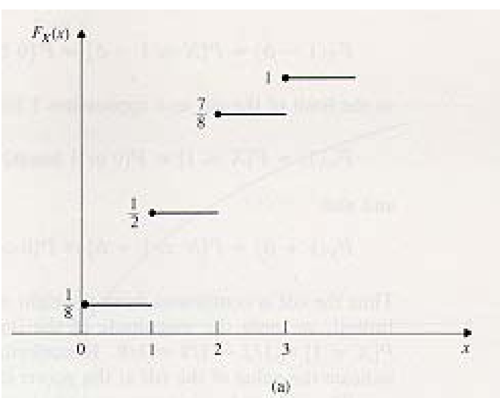
7. $P[b - \varepsilon < X \le b] = F_x(b) - F_x(b^-)$

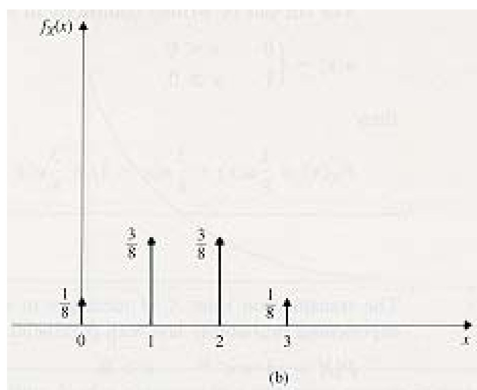
   If $\varepsilon \to 0$ then $P[X = b] = F_x(b) - F_x(b^-)$
   and if cdf is continuous at $x = b$, then $\{X=b\}$ has probability zero.

8. Correlary 1: $P[X > x] = 1 - F_x(x)$

Ex: 3.4 and 3.5 Tossing 3 coins and {x} = {# of heads}

cdf and **pdf**





Near x = 1 let $\delta > 0$, small then

$F_x(1- \delta) = P[X \le 1 - \delta] = P\{0 \text{ heads}\} = 1/8$

But: $F_x(1) = P[X \le 1] = P\{ 0 \text{ or } 1 \text{ heads}\} = 1/8 + 3/8 = 1/2$

and $F_x(1+ \delta) = P[X \le 1 + \delta] = P\{ 0 \text{ or } 1 \text{ heads}\} = 1/2$

cdf can be written in terms of unit step functions when there are discontinuities:

$$F_x(x) = (1/8)u(x) + (3/8)u(x-1) + (3/8)u(x-2) + (1/8)u(x-3)$$

pdf can be written in terms of $\delta(\cdot)$ function for discrete prob. events:

$$f_x(x) = (1/8)\,\delta(x) + (3/8)\,\delta(x-1) + (3/8)\,\delta(x-2) + (1/8)\,\delta(x-3)$$

and

$$P[1 < X \le 2] = \int_{1^+}^{2} f_X(x)dx = \frac{3}{8} \qquad P[2 \le X < 3] = \int_{2}^{3^-} f_X(x)dx = \frac{3}{8}$$

Ex: 3.5 Transmission time X in a communication system obeys

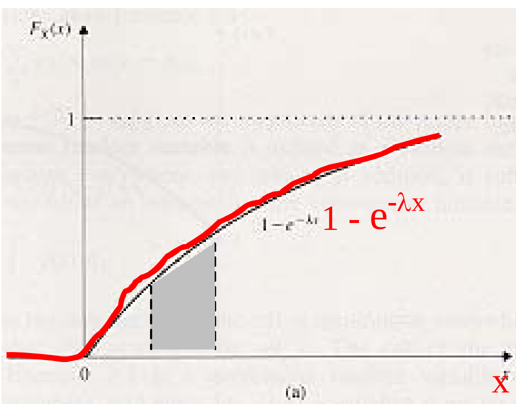
$$P(X > x) = e^{-\lambda x} \quad x > 0 \text{ and } \lambda = rate = 1/T$$

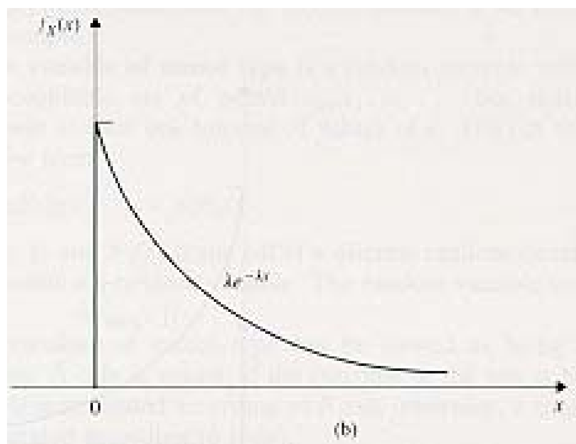
cdf: $$F_X(x) = P[X \le x] = 1 - P[X > x] = \begin{cases} 1 - e^{-\lambda x} & if\ x > 0 \\ 0 & if\ x < 0 \end{cases}$$

Find:

$$P[T < X \le 2T] = (1 - e^{-2}) - (1 - e^{-1}) = e^{-1} - e^{-2} \approx 0.233$$

$$F'(x) = \begin{cases} \lambda.e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$ and pdf: $f_x(x)$





Discrete r.v. are described by prob. mass function (pmf) of X as the set of probabilities

$$p_X(x) = P[X = x_k] \text{ in } S_x.$$

cdf for discrete r.v.: $$F_X(x) = \sum_k p_X(x) u(x - x_k)$$

Continuous r.v. is a r.v. with a continuous cdf and the cdf is equal to the area under the pdf curve upto the point x: $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$

Mixed r.v. has a cdf with jumps on a countable set of points but also increase continuously over at least on one interval.

cdf : $$F_X(x) = pF_1(x) + (1-p)F_2(x) \qquad 0 < p < 1$$

Cdf of discrete r.v. cdf of cont. r.v.

R.V. Examples: Discrete: p. 100

TABLE 3.1
Discrete Random Variables

Bernoulli Random Variable

$S_X = \{0, 1\}$

$p_0 = q = 1 - p \qquad p_1 = p \qquad 0 \le p \le 1$

$E[X] = p \qquad \text{VAR}[X] = p(1-p)$

$G_X(z) = (q + pz)$

Remarks: The Bernoulli random variable is the value of the indicator function I_A for some event A; $X = 1$ if A occurs and $X = 0$ otherwise.

$I_A(x) = \begin{cases} 0 & \xi \notin A \\ 1 & \xi \in A \end{cases}$

P[A] = p success

Binomial Random Variable

$S_x = \{0, 1, \ldots, n\}$

$p_k = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n$

$E[X] = np \qquad \text{VAR}[X] = np(1-p)$

$G_X(z) = (q + pz)^n$

Remarks: X is the number of successes in n Bernoulli trials and hence the sum of n independent, identically distributed Bernoulli random variables.

Let X be # of times A occurs in n trials. If I_j is indicator fn. for A in jth trial, then $X = I_1+I_2+\ldots+I_n$ and

$P[X = k] = p_k$

Geometric Random Variable

First Version: $S_x = \{0, 1, 2, \ldots\}$

$p_k = p(1-p)^k \qquad k = 0, 1, \ldots$

$E[X] = \frac{1-p}{p} \qquad \text{VAR}[X] = \frac{1-p}{p^2}$

$G_X(z) = \frac{p}{1-qz}$

pmf: $P[M = k] = p_k$ which decays geometrically with k where

p = P[A] is prob. success in each Bernoulli trial. (see fig 3.9)

Remarks: X is the number of failures before the first success in a sequence of independent Bernoulli trials.

The geometric random variable is the only discrete random variable with the memoryless property.

Second Version: $S_{X'} = \{1, 2, \ldots\}$

$p_k = p(1-p)^{k-1} \qquad k = 1, 2, \ldots$

$E[X'] = \frac{1}{p} \qquad \text{VAR}[X'] = \frac{1-p}{p^2}$

$G_{X'}(z) = \frac{pz}{1-qz}$

Remarks: $X' = X + 1$ is the number of trials until the first success in a sequence of independent Bernoulli trials.

Negative Binomial Random Variable

$S_X = \{r, r+1, \ldots\}$, where r is a positive integer

$p_k = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad k = r, r+1, \ldots$

$E[X] = \frac{r}{p} \qquad \text{VAR}[X] = \frac{r(1-p)}{p^2}$

$G_X(z) = \left(\frac{pz}{1-qz}\right)^r$

Remarks: X is the number of trials until the rth success in a sequence of independent Bernoulli trials.

Poisson Random Variable

$S_X = \{0, 1, 2, \ldots\}$

$p_k = \frac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, \ldots \text{ and } \alpha > 0$

$E[X] = \alpha \qquad \text{VAR}[X] = \alpha$

$G_X(z) = e^{\alpha(z-1)}$

Remarks: X is the number of events that occur in one time unit when the time between events is exponentially distributed with mean $1/\alpha$.

# of occurrences of an event in a certain time period as in counts of radioactive substances, counts of demands for tel. networks connections.

pmf: $$P[N = k] = \frac{\alpha^k}{k!} e^{-\alpha} \qquad for \quad k = 0, 1, \ldots, \quad \alpha > 0$$

α: ave. # of occurrances in a specified time unit (fig 3.10 for pmf)

Remarks on Discrete Distributions of Table 3.1

1) cdf of geometric r.v.

$$P[M \le k] = \sum_{j=1}^{k} pq^{j-1} = p\sum_{l=0}^{k-1} q^{l} = p\frac{1-q^{k}}{1-q} = 1-q^{k}$$

$$P[N = k] = P[M \ge k+1] = (1-p)^{k}\, p \quad k = 0,1,2,...$$

Let N = M-1 # of failures before a success occurs, then
Geometric r.v. satisfies **memoryless** property:

$$P[M \ge k + j \mid m > j] = P[M \ge k] \quad \textit{for all } j,k > 1$$

Thus, each time a failure occurs, the system forgets and begins anew as if it were performing first trial. It occurs in queuing system models.

2) pmf of Poisson r.v. sums to 1:

$$\sum_{k=0}^{\infty} \frac{\alpha^{k}}{k!} e^{-\alpha} = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^{k}}{k!} = e^{-\alpha} e^{\alpha} = 1$$

$$p_k = \binom{n}{k} p^{k}(1-p)^{n-k} \approx \frac{\alpha^{k}}{k!} e^{-\alpha} \quad \textit{for } k = 0,1,...$$

Law of large numbers for Bernoulli trials: If n is large and p > 0 small, then for $\alpha \equiv np$

Ex: 3.11 Given $p_e = 10^{-3}$ Find a packet of 1000 bits that has ≥ 5 errors

Since this is a Bernoulli trial with n = 1000, p = 10^{-3} Poisson approximation:

$$P[N \ge 5] = 1 - P[N < 5]$$

$$= 1 - \sum_{k=0}^{4} \frac{\alpha^{k}}{k!} e^{-\alpha} = 1 - e^{-1}\{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\}$$

$$= 0.00366$$

Since $\alpha \equiv np$
$= 1000(10^{-3})$

Table 3.2 Continuous r.v. p.101

Uniform Random Variable

$S_X = [a, b]$

$$f_X(x) = \frac{1}{b-a} \qquad a \le x \le b$$

$$E[X] = \frac{a+b}{2} \qquad \text{VAR}[X] = \frac{(b-a)^2}{12}$$

$$\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$$

Exponential Random Variable

$S_X = [0, \infty)$

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \ge 0 \qquad \text{and } \lambda > 0$$

$$E[X] = \frac{1}{\lambda} \qquad \text{VAR}[X] = \frac{1}{\lambda^2}$$

$$\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$$

Remarks: The exponential random variable is the only continuous random variable with the memoryless property.

Gaussian (Normal) Random Variable

$S_X = (-\infty, +\infty)$

$$f_X(x) = \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} \qquad -\infty < x < +\infty \quad \text{and} \quad \sigma > 0$$

$$E[X] = m \qquad \text{VAR}[X] = \sigma^2$$

$$\Phi_X(\omega) = e^{jm\omega - \sigma^2\omega^2/2}$$

Remarks: Under a wide range of conditions, X can be used to approximate the sum of a large number of independent random variables.

Cumulative Distribution for Exponential Function

$$\text{cdf } F_x(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-e^{-\lambda x} & \text{if } x \ge 0 \end{cases}$$

Table 3.2 Continued

Gamma Function:

$$\Gamma(z) = \int_0^\infty x^{z-1}.e^{-x}dx \quad \text{if } z > 0$$

with properties:

$\Gamma(0.5) = \sqrt{\pi}$

$\Gamma(z+1) = z\Gamma(z) \quad \text{for} \quad z > 0$

$\Gamma(m+1) = m!$

Gamma Random Variable

$S_X = (0, +\infty)$

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} \qquad x > 0 \quad \text{and} \quad \alpha > 0, \lambda > 0$$

where $\Gamma(z)$ is the gamma function (Eq. 3.46).

$E[X] = \alpha/\lambda \qquad \text{VAR}[X] = \alpha/\lambda^2$

$$\Phi_X(\omega) = \frac{1}{(1 - j\omega/\lambda)^\alpha}$$

Special Cases of Gamma Random Variable

m-Erlang Random Variable: $\alpha = m$, a positive integer

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{m-1}}{(m-1)!} \qquad x > 0$$

$$\Phi_X(\omega) = \left(\frac{\lambda}{\lambda - j\omega}\right)^m$$

Remarks: An m-Erlang random variable is obtained by adding m independent exponentially distributed random variables with parameter λ.

Chi-Square Random Variable with k degrees of freedom: $\alpha = k/2$, k a positive integer and $\lambda = \frac{1}{2}$

$$f_X(x) = \frac{x^{(k-2)/2}e^{-x/2}}{2^{k/2}\Gamma(k/2)} \qquad x > 0$$

$$\Phi_X(\omega) = \left(\frac{1}{1 - j2\omega}\right)^{k/2}$$

Remarks: The sum of k mutually independent, squared zero-mean unit-variance Gaussian random variables is a chi-square random variable with k degrees of freedom.

Rayleigh Random Variable

$S_X = [0, \infty)$

$$f_X(x) = \frac{x}{\alpha^2}e^{-x^2/2\alpha^2} \qquad x \geq 0 \qquad \alpha > 0$$

$E[X] = \alpha\sqrt{\pi/2} \qquad \text{VAR}[X] = (2 - \pi/2)\alpha^2$

Cauchy Random Variable

$S_X = (-\infty, \infty)$

$$f_X(x) = \frac{\alpha/\pi}{x^2 + \alpha^2} \qquad -\infty < x < \infty \qquad \alpha > 0$$

Mean and variance do not exist.

$\Phi_X(\omega) = e^{-\alpha|\omega|}$

Laplacian Random Variable

$S_X = (-\infty, \infty)$

$$f_X(x) = \frac{\alpha}{2}e^{-\alpha|x|} \qquad -\infty < x < \infty \qquad \alpha > 0$$

$E[X] = 0 \qquad \text{VAR}[X] = 2/\alpha^2$

$$\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$$

Remarks on Continuous r.v.

1. See Fig 3.12 as a limiting behavior for cdf of a discrete r.v. $\rightarrow$ uniform cont. r.v.

2. Exp. r.v. is a limiting form of geometric r.v. (Fig. 3.10.a)
   # of subintervals until the occurrence of an event X = MT/n
   where M: geo. r.v. , n: #of Bernoulli trials, T: time interval

$$P[M > t] = P[M > n\frac{t}{T}] = [1-p]^{\frac{nt}{T}} = [(1-\frac{\alpha}{n})^{n}]^{\frac{t}{T}} \rightarrow e^{\frac{-\alpha t}{T}} \quad as\ n \rightarrow \infty$$

3. Exp. r.v. satisfies the memoryless property: $P[X > t+h \mid X > t] = P[X > h]$

Proof:

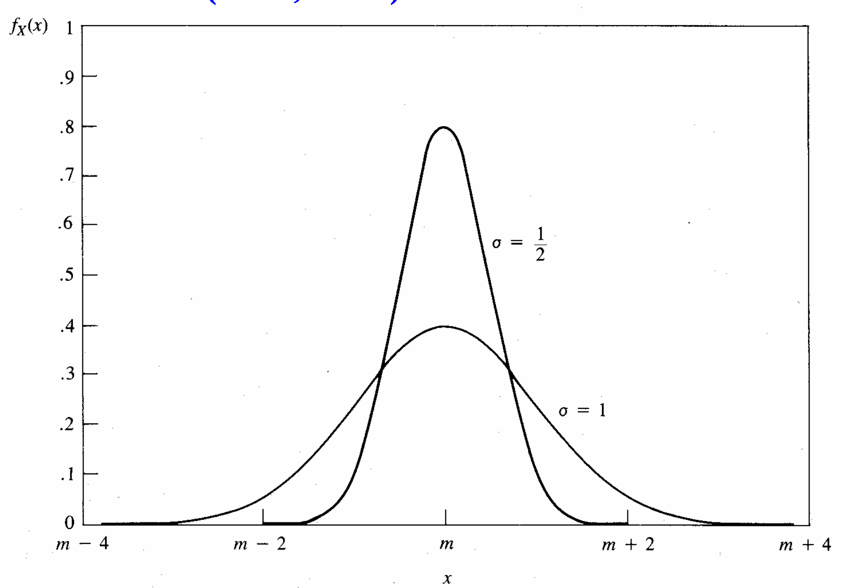
$$P[X > t+h \mid X > t] = \frac{P[(X. > t+h) \cap (X > t)]}{P[X > t]} \quad for\ h > 0$$

$$= \frac{P[X > .t+h]}{P[X > t]} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = P[X > h]$$

4. cdf of Gaussian r.v.: If x' is the dummy integration variable:

$$P[X \le x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{\frac{-(x'-m)^2}{2\sigma^2}} dx' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\left(\frac{x-m}{\sigma}\right)} e^{\frac{-t^2}{2}} dt = \Phi\left(\frac{x-m}{\sigma}\right)$$

Standard Gaussian r.v. ***N*(m = 0, σ^2 = 1)**



5. Q-Function

$$Q(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{\frac{-t^2}{2}} dt$$

Tail area of the pdf.

with $Q(0)=\frac{1}{2}$ and $Q(-x)=1-Q(x)$ (Study Table 3.3, 3.4)

Table: 3.4 "Value of x for which Q(x)=10^{-k}"

Approximation for the Q-function:

$$Q(x)\approx\left[\frac{1}{(1-a)x+a\sqrt{x^2+b}}\right]\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}} \qquad a=\frac{1}{\pi};\quad b=2\pi$$

Ex: 3.15 Signal in : V Volts $\alpha = 10^{-2}$

Signal out: $Y = \alpha V + N$ $N = N(m=0, \sigma^2 = 4)$ Gaussian

Find V such that $P[Y<0] = 10^{-6}$

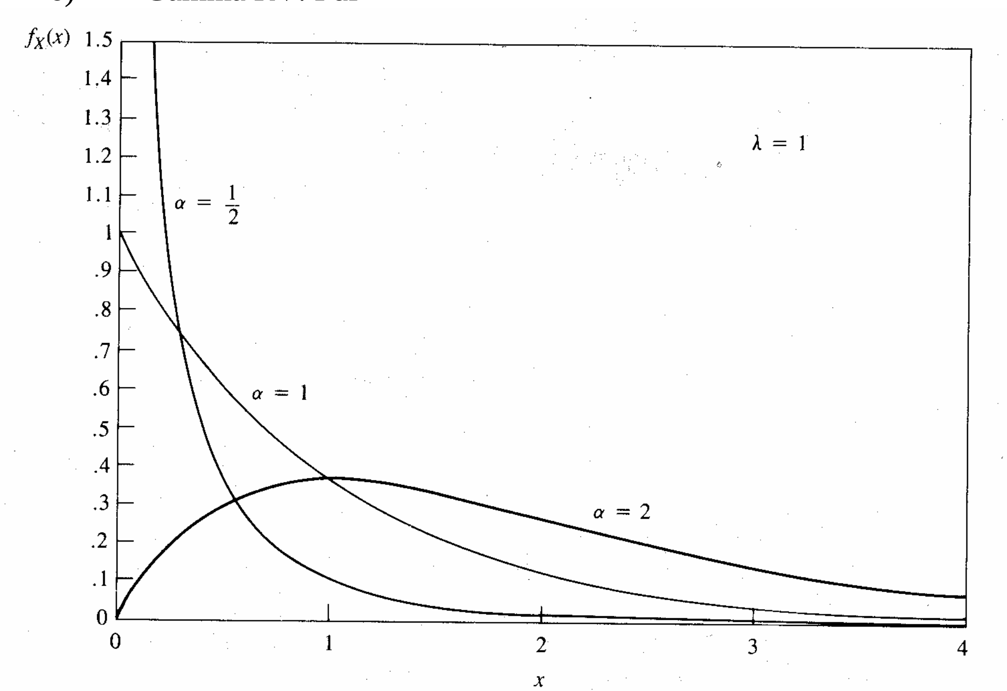
$$P[Y<0]=P[\alpha V+N<0]=P[N<-\alpha V]$$

$$=\Phi\left(\frac{-\alpha V}{\sigma}\right)=Q\left(\frac{\alpha V}{\sigma}\right)=10^{-6}$$

$$\frac{\alpha V}{\sigma}=\frac{\left(10^{-2}\right)V}{2}\Rightarrow \quad 10^{-6}\rightarrow k=4.753$$

$$V=(4.753)\frac{2}{10^{-2}}=950.6 \quad volts$$

6) Gamma RV. Pdf



Exponential case: α = 1

Chi-square case:
λ = 1/2
and α = k/2
with k>0 integer

Functions of Single RandomVariable

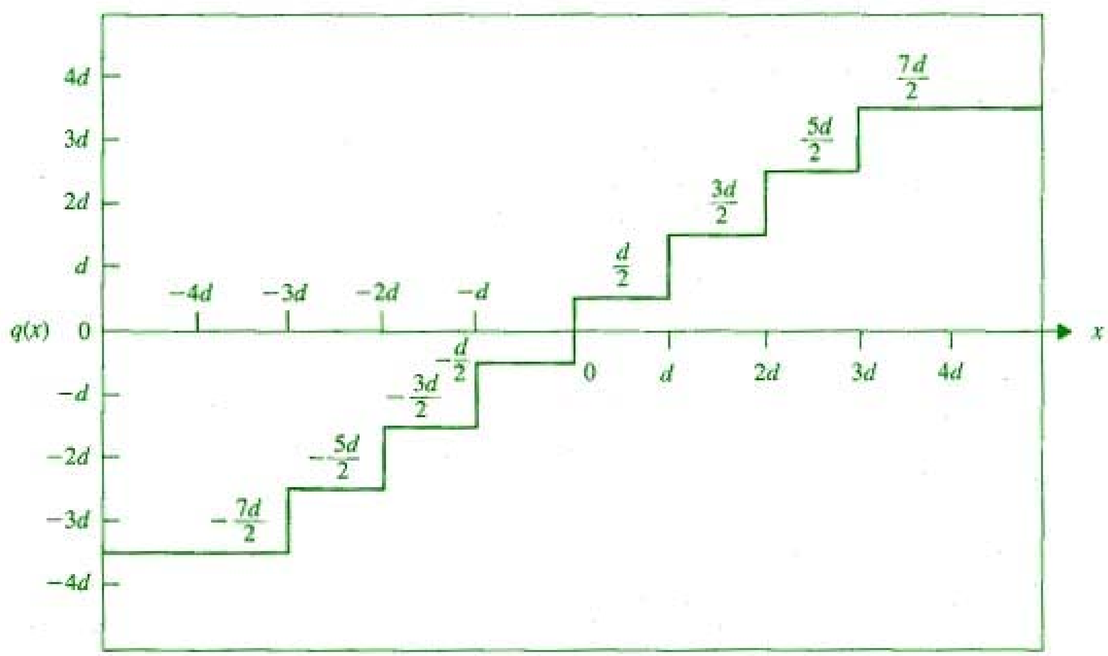
Define a new r.v. such that: $Y = q(X)$

Task: Find pdf and cdf of Y in terms of those from r.v. X.

Ex: 3.19 Uniform quantizer: (8-level)

X : input signal to the quantizer and Y = q(x) : quantized output
$S_y = \{-3.5d, -2.5d, -1.5d, -0.5d, 0.5d, 1.5d, 2.5d, 3.5d\}$

Rule: All points in the interval (0,d) are mapped to: q(x)=d/2



PROCESSING RULE: P[Y in C] = P[q(x) in C] = P[X in B], where C and B are equivalent events in S_y, S_x

Ex: 3.22 Quantizing Speech samples into 3-bits uniform quantizer

Given X is uniform in [-4d, 4d] and Y = q(X). Find pmf for the quantized signal Y. The event $\{Y = q; q \in S_y\}$ is equivalent to $\{X \text{ in } I_q\}$ where I_q is a group of samples mapped into a representation point q.

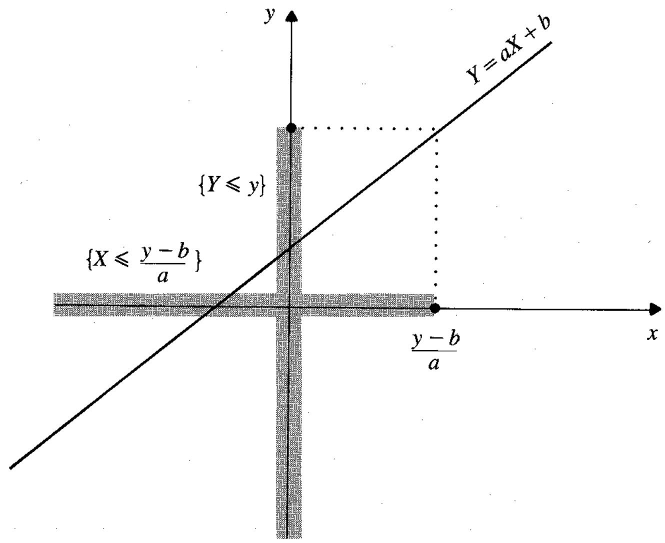
pmf for Y: $P[y = q] = \int_{I_q} f_X(t)dt = \frac{1}{8}$ Note: 8 outputs are equiprobable.

Ex: 3.23 *Let* $Y = aX + b$ with $F_X(x)$ and $a \neq 0$

Find: $F_Y(y)$

$\{Y \le y\}$ occurs when

$A = \{aX + b \le y\}$ occurs.



1. If $a>0$ then $A = \left\{X \le \frac{y-b}{a}\right\}$ and the cdf is written as:

$$F_Y(y) = P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right) \quad for\ a>0$$

2. If $a < 0$ then $A = \left\{X > \frac{y-b}{a}\right\}$

$$F_Y(y) = P\left[X \ge \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right)$$

pdf: Using the derivative rule: $\frac{dF}{dy} = \frac{dF}{du}\frac{du}{dy}$ and $u = \frac{y-b}{a}$; we obtain:

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & if\ a>0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & if\ a<0 \end{cases} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

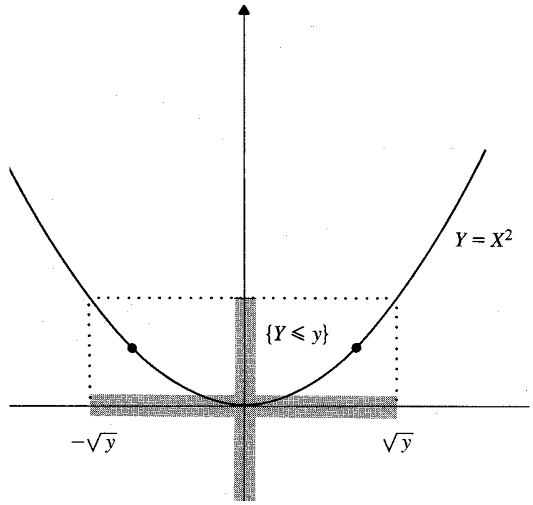
Ex: 3.24 Given X with a Gaussian pdf: $N(m,\sigma^2)$ and $Y = aX + b$. Find $f_y(y)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} \quad -\infty < x < \infty \ and \ f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{|a\sigma|} e^{-(y-b-am)^2/2(a\sigma)^2}$$

It is also a Gaussian r.v. with mean $b+am$ and st.dev. $|a|\sigma$

Ex: 3.25 Given: $Y = X^2$; find cdf and pdf of Y

$\{Y \le y\}$ is equivalent to saying: $X^2 \le y$ and $-\sqrt{y} \le X \le \sqrt{y}$ *for* $y > 0$.



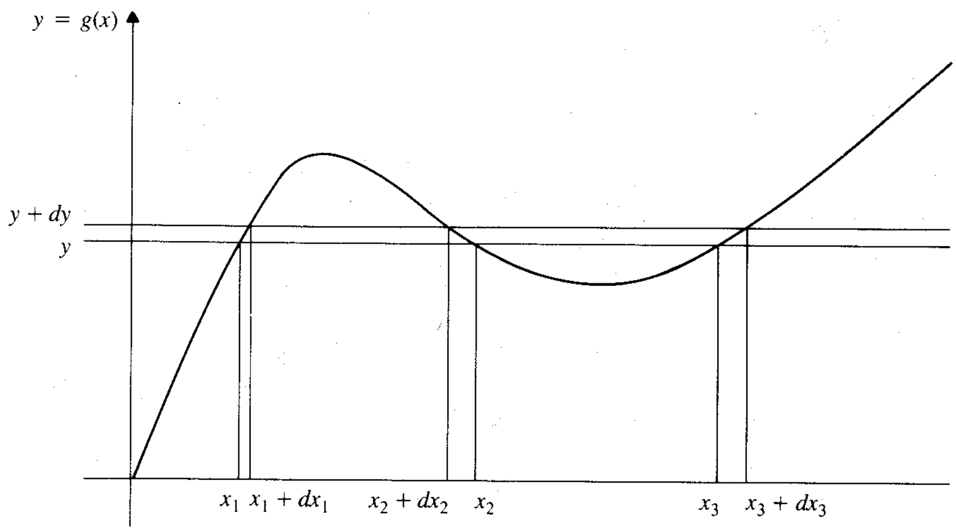
This results in a cdf:

$$F_Y(y) = \begin{cases} F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) & \text{if } y > 0 \\ 0 & \text{if } y \le 0 \end{cases}$$

$$\frac{dF_Y}{dy} \Rightarrow f_Y(y) = \begin{cases} \dfrac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} - \dfrac{f_X\left(-\sqrt{y}\right)}{-2\sqrt{y}} & y > 0 \\ 0 & y \le 0 \end{cases}$$

Multiple Roots Case: Y = g(X) $C_y = \{y<Y\le y+dy\}$

In this case, g(x) = y has multiple solutions: $x_1, x_2, x_3, \ldots$



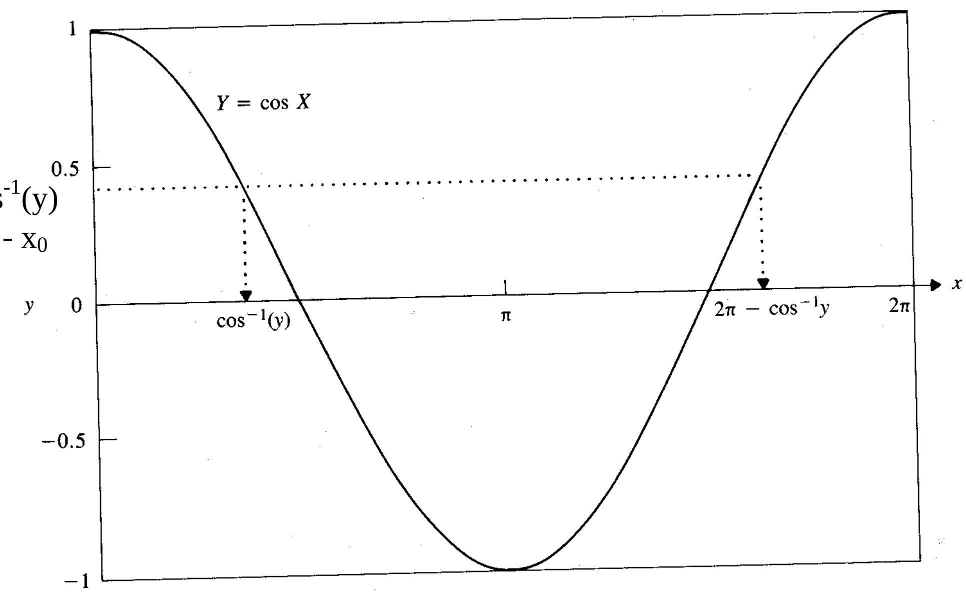
B_y has equivalent events: $B_y = \{x_1<X<x_1+dx_1 \cup \ldots \cup x_3<X<x_3+dx_3\}$

$$P[C_y] = f_y(y)|dy| \Rightarrow P[B_y] = f_X(x_1)|dx_1| + f_X(x_2)|dx_2| + f_X(x_3)|dx_3|$$

Ex: 3.28: Samples of a sinusoid. Let Y = cos(x) and X is uniform in (0, 2π). Find pdf and cdf of Y?

Since X is uniform we know that: $f_X(x) = \begin{cases} \frac{1}{2\pi} & 0<x<2\pi \\ 0 & Otherwise \end{cases}$

When 0 < x < 2π
We have,
-1 < y < 1
y = cos(x) ⇒ $x_0 = \cos^{-1}(y)$
$x_1 = 2\pi - x_0$



But

$$\frac{dy}{dx}|_{x_0} = -\sin(x_0) = -\sin(\cos^{-1}(y)) = -\sqrt{1-y^2}$$

which results in a **pfd**:

$$f_Y(y) = \frac{1/2\pi}{\sqrt{1-y^2}} + \frac{1/2\pi}{\sqrt{1-(-y)^2}} = \frac{1/\pi}{\sqrt{1-y^2}} \quad for\ -1<y<1$$

cdf becomes an arcsine distribution:

$$F_Y(y) = \int_{-\infty}^{y} f_y(y')dy' = \begin{cases} 0 & if\ y<-1 \\ \frac{1}{2}+\frac{\sin^{-1} y}{\pi} & if\ -1\le y\le 1 \\ 1 & if\ y>1 \end{cases}$$

Expected values:

Expected Value (mean): $E[X]=\int_{\infty} t.f_X(t)dt \qquad E[X]=\sum_{\infty} x_k.P_X(x_k)$ The mean exists if

$$E[X]=\int_{-\infty}^{\infty}|t|\,f_X(t)dt<\infty \quad or \quad E[|X|]=\sum_{k}|x_k|P_X(x_k)<\infty$$

Top curve varies around 5.0 and wide spread

Bottom curve varies around 0.0 and little spread

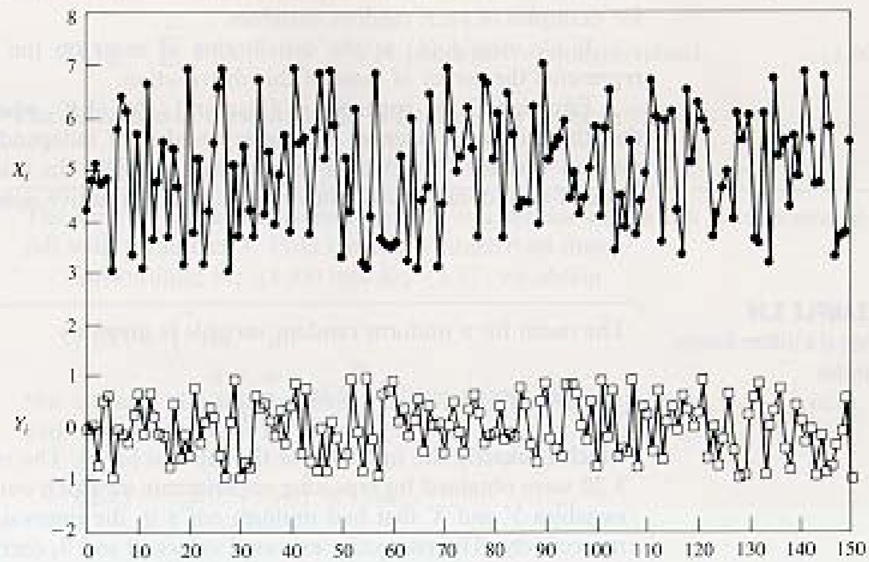


FIGURE 3.20 The graphs show 150 repetitions of the experiments yielding the random variable X and the random variable Y. It is clear that X takes on values centered about the value 5 while Y takes on values centered about 0. It is also clear that X is more spread out than Y.

Ex: 3.29 Uniform r.v. – pdf/mean

$$f_X(x)=\begin{cases}\dfrac{1}{b-a} & a\le x\le b\\ 0 & Otherwise\end{cases}$$

$$E[X]=\frac{1}{b-a}\int_a^b t\,dt=\frac{1}{b-a}\frac{t^2}{2}\Big|_a^b=\frac{b^2-a^2}{2(b-a)}=\frac{a+b}{2}$$

Notes: If X ≥ 0, then

$$E[X]=\int_0^{\infty}(1-F_X(t)dt \qquad if\;\; x:continuous$$

$$E[X]=\sum_{k=0}^{\infty}P[X>k] \qquad if\;\; k:discrete.$$

Ex: 3.31 Inter-arrival time average. f_x(x) is exponential with λ and 1/λ seconds per customer:

$$E[X] = \int_0^\infty t\lambda e^{-\lambda t}dt = -te^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}dt$$

When we use the integration by parts with the terminology:

$$\int u dv = uv - \int v du \qquad u = t, \quad dv = \lambda e^{-\lambda t}dt$$

which results with the expected value:

$$E[X] = \lim_{t\to\infty} te^{-\lambda t} - 0 + \frac{-e^{-\lambda t}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

Variance and standard deviation of X:

$$\sigma_X^2 = VAR[X] = E\big[X - E[X]\big]^2$$

$$\sigma_X = STD[X] = \sqrt{VAR[X]}$$

In practice we use a slightly different version for the variance expression:

$$\sigma_X^2 = E\left[X^2 - 2XE[X] + E[X]^2\right] = E[X^2] - E[X]^2$$

Ex: 3.36 Variance of uniform r.v. X for a ≤ x ≤ b

$$\sigma_X^2 = \int_a^b \frac{1}{b-a}\left(x - \frac{a+b}{2}\right)^2 dx = \left(\frac{1}{b-a}\right).\int_{-(b-a)/2}^{(b-a)/2} y^2 dy$$

$$= \frac{(b-a)^3}{12(b-a)} = \frac{(b-a)^2}{12}$$

In the above integral we have used a change of variable: $y = x - \frac{a+b}{2} \quad dy = dx$

Ex: 3.38 Variance of a Gaussian r.v.

$$\frac{1}{\sqrt{2\pi}\,\sigma_X}.\left(\int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma_X^2}}dx\right) = 1 \qquad \Rightarrow \qquad \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma_X^2}}dx = \sqrt{2\pi}\,\sigma_X$$

which can be re-written by differentiating with respect to: σ_X

$$\int_{-\infty}^{\infty} \frac{(x-m)^2}{\sigma_X^3} e^{-\frac{(x-m)^2}{2\sigma_X^2}}dx = \sqrt{2\pi}$$

Let us re-arrange this result to obtain the expression for the variance:

$$\sigma_X^2 = \frac{1}{\sqrt{2\pi}\,\sigma_X}.\left(\int_{-\infty}^{\infty} (x-m)^2 e^{-\frac{(x-m)^2}{2\sigma_X^2}}dx\right)$$

Notes: 1) VAR[C] = 0
2) VAR[X+C] = VAR[X]
3) VAR[CX] = C^2VAR[X]
4) $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx \qquad n^{th}\ moment\ of\ r.v.\ X$

Ex: 3.39 Uniform Quantizer: X ⇒ q(X) from 2^R levels (R-bit) with a Quantizing Noise: Z = X-q(X)

FIGURE 3.22
The uniform quantizer error for the input x is $x - q(x)$.

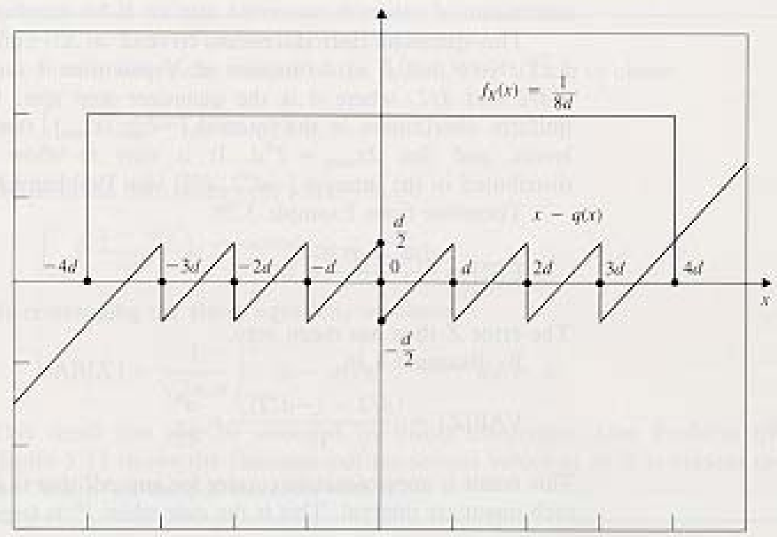
d: step size

$-d/2 \le Z \le d/2$

and

$-x_{max} \le X \le x_{max}$

$\Rightarrow\ 2x_{max} = 2^R d$



Using

$$E[Z] = \frac{d/2 - d/2}{2} = 0 \quad \text{(Zero-mean)} \quad \text{and} \quad VAR[Z] = \frac{[d/2 - (-d/2)]^2}{12} = \frac{d^2}{12}$$

Recall:

X – Z = q(x)

Signal-Quantizing Noise Ratio:

$$SNR = \frac{VAR[X]}{VAR[Z]} = \frac{VAR[X]}{d^2/12} = 3\frac{VAR[X]}{x_{max}^2} 2^{2R}$$

However, almost always, we express the SNR in decibels (dB)

$$SNR_{dB} = 10\log_{10} SNR \approx 6R - 7.3dB$$

In the last approximation, we have used the industry standard: $x_{max} = 4\sigma_X$ known as the 4-sigma loading condition.

Each additional bit doubles the number of quantizer levels and the step size d is reduced by a factor of 2 ⇒ VAR[Z] will be reduced by 2^2 = 4

MARKOV and CHEBYSHEV INEQUALITIES

Let X ≥ 0 and mean = E[X], then the Markov Inequality is written by

$$P[X \geq a] \leq \frac{E[X]}{a} \qquad for\ x \geq a$$

$$E[X] = \int_0^a t\, f_X(t)dt + \int_a^\infty t\, f_X(t)dt \geq \int_a^\infty t\, f_X(t)dt \geq \int_a^\infty a\, f_X(t)dt$$

which result in:

$$E[X] \geq a\int_a^\infty f_X(t)dt = aP[X \geq a]$$

$$\Rightarrow \quad P[X \geq a] \leq \frac{E[X]}{a}$$

Let X have a mean m and *VAR[X]* = σ_X^2 then

$$P[|X - m| \geq a] = P[-a \geq X - m \geq a]$$

$$= P[-a + m \geq X \geq a + m] \leq \frac{\sigma_X^2}{a^2}$$

Chebyshev Inequality

Ex: 3.41 For a response time = 15 s.
St. dev. of resp. time = 3 s.
Find prob. that the response time > 5 s. from mean

m = 15 s. σ = 3 s. a = 5

$$P[|X - 15| \geq 5] \leq \frac{9}{25} = 0.36$$

(Skip 3.8 Fit of Distr. Of Data)

Characteristic and Probability Generating Functions

Characteristic function:

$$\Phi_X(w) = E\left[e^{jwX}\right] = \int_{-\infty}^{\infty} f_X(x)\, e^{jwX}\, dx$$

1) $\Phi_X(w)$ is the expected value of a fn of X: $g(X) = e^{jwX}$ g(x) = e^{jwX}
2) $\Phi_X(w)$ is the Fourier Tx. of pdf $f_x(x)$.
   In which case, the inverse Fourier Tx:

$$f_X(x) = \Im^{-1}\{\Phi_X(w)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(w)\, e^{-jwX}\, dw$$

If X is a discrete r.v. then

$$\Phi_X(w) = \sum_k p_X(x_k)\, e^{jwX_k}$$

Furthermore, if X_k is integer then:

$$\Phi_X(w) = \sum_{k=-\infty}^{\infty} p_X(k)\, e^{jwk}$$

which is the **Fourier transform** of the probability mass function $p(k)$.

Inverse Fourier Tx.:

$$p_X(k) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_X(w)\, e^{-jwk}\, dw \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Ex: 3.47 $\Phi_X(w)$ for the exponential r.v. X:

$$f_X(x) = \begin{cases} \lambda . e^{-\lambda . x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\Phi_X(w) = \int_0^{\infty} \lambda\, e^{-\lambda x}\, e^{jwx} dx = \lambda \int_0^{\infty} e^{-(\lambda - jw)x} dx$$

$$\Phi_X(w) = \frac{\lambda}{\lambda - jw}$$

Ex: 3.48 $\Phi_x(w)$ for geometric r.v.

$$\Phi_X(w) = \sum_{k=0}^{\infty} pq^k\, e^{jwk} = p\sum_{k=0}^{\infty}\left(q\, e^{jw}\right)^k = p\,\frac{1}{1 - q\, e^{jw}}$$

Moment Generating Function:

$$E\left[X^n\right] = \frac{1}{j^n} \frac{d^n}{dw^n} \Phi_X(w) \Big|_{w=0}$$

Proof: Expand characteristic function in a power series expansion:

$$\Phi_X(w) = \int_{-\infty}^{\infty} f_X(x) \left[1 + jwx + \frac{(jwx)^2}{2!} + \frac{(jwx)^3}{3!} + \dots \right] dx$$

$$= 1 + jw \int_{-\infty}^{\infty} x f_X(x) dx + \frac{(jw)^2}{2!} \int_{-\infty}^{\infty} x^2 f_X(x) dx + \dots$$

$$= 1 + jwE[X] + \frac{(jw)^2}{2!} E[X^2] + \dots + \frac{(jw)^n}{n!} E[X^n] + \dots$$

If we differentiate once wrt to w and set w = 0

$$\frac{d}{dw} \Phi_X(w) \Big|_{w=0} = jE[X] \quad \Rightarrow \quad E[X] = \frac{1}{j} \frac{d}{dw} \Phi_X(w) \Big|_{w=0}$$

Differentiate twice and set w = 0 yields

$$\frac{d^2}{dw^2} \Phi_X(w) \Big|_{w=0} = -E[X^2]$$

Similarly,

$$\frac{d^n}{dw^n} \Phi_X(w) \Big|_{w=0} = j^n E[X^n]$$

Ex: 3.49 Exponential pdf and char. fn: $\Phi_X(w) = \dfrac{\lambda}{\lambda - jw}$

Let us differentiate it once:

$$\Phi'_X(w) = \frac{\lambda j}{(\lambda - jw)^2}$$

We obtain:

$$E[X] = \frac{\Phi'_X(0)}{j} = \frac{1}{\lambda}$$

Similarly, one more differentiation results in

$$\Phi''_X(w) = \frac{-2\lambda}{(\lambda - jw)^3}$$

$$E[X^2] = \frac{\Phi''_X(0)}{j^2} = \frac{-2\lambda}{-\lambda^3} = \frac{2}{\lambda^2}$$

Using these two statistics we compute the variance:

$$\sigma_X^2 = VAR[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

pdf and pmf Generating Functions:

PDF Generating Function:

1) $G_N(z)$ is the expected value of a function of N : $g(N) = z^N$
2) $G_N(z)$ is the z-Tx. of pmf and $\Phi_N(w) = G_N(e^{jw})$

Similarly, pmf gen. fn:

$$p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z)\Big|_{z=0}$$

with statistics:

$$E[N] = \frac{d}{dz} G_N(z)\Big|_{z=1} = \sum_{k=0}^{\infty} p_N(k) k z^{k-1}\Big|_{z=1} = \sum_{k=0}^{\infty} k\, p_N(k) = E[N]$$

and

$$\frac{d^2}{dz^2} G_N(z)\Big|_{z=1} = \sum_{k=0}^{\infty} p_N(k) k(k-1) z^{k-1}\Big|_{z=1} = \sum_{k=0}^{\infty} k(k-1)\, p_N(k)$$

$$= E[N(N-1)] = E[N^2] - E[N]$$

Furthermore,

$$E[N] = G'_N(1)$$

$$VAR[N] = G''_N(1) + G'_N(1) - [G'_N(1)]^2$$

Ex: 3.50 Poisson r.v. with parameter α :

$$G_N(z) = \sum_{k=0}^{\infty} \left[\frac{\alpha^k}{k!} e^{-\alpha}\right] z^k$$

$$G_N(z) = e^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!} = e^{-\alpha} e^{\alpha Z} = e^{\alpha(Z-1)}$$

Taking the first two derivatives: $G'_N(z) = \alpha\, e^{\alpha(z-1)}$; and $G''_N(z) = \alpha^2 e^{\alpha(z-1)}$ yields the answer:

$$E[N] = \alpha \quad \textit{and} \quad VAR[N] = \sigma_N^2 = \alpha^2 + \alpha - \alpha^2 = \alpha$$

Laplace Tx of pdf (Nonnegative continuous r.v.)

$$X^*(s) = \int_0^{\infty} f_X(x)\, e^{-sx}\, dx = E\left[e^{-sx}\right] \quad \text{and} \quad E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s)\Big|_{s=0}$$

Ex: 3.51 Laplace Tx method on Gamma pdf:

$$X^*(s) = \int_0^{\infty} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} e^{-sx}\, dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\lambda+s)x}\, dx$$

Using the following substitution of variable: $y = \lambda + s$

$$X^*(s) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \frac{1}{(\lambda+s)^{\alpha}} \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \frac{1}{(\lambda+s)^{\alpha}} \Gamma(\alpha) = \frac{\lambda^{\alpha}}{(\lambda+s)^{\alpha}}$$

The expected value and the mean-square value:

$$E[X] = -\frac{d}{ds} \frac{\lambda^{\alpha}}{(\lambda+s)^{\alpha}}\Big|_{s=0} = \frac{\alpha\lambda^{\alpha}}{(\lambda+s)^{\alpha+1}}\Big|_{s=0} = \frac{\alpha}{\lambda}$$

$$E[X^2] = \frac{d^2}{ds^2} \frac{\lambda^{\alpha}}{(\lambda+s)^{\alpha}}\Big|_{s=0} = \frac{\alpha(\alpha+1)\lambda^{\alpha}}{(\lambda+s)^{\alpha+2}}\Big|_{s=0} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

Finally, the variance:

$$\sigma_X^2 = E[X^2] - E[X]^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

(Skip 3.10, 11 and 12)

#3.1 Urn contains 90 -- \$1; 9 -- \$5; 1 -- \$50 Let X be denomination of bill
a) Describe space, S. Specify probability of events

The sample space has 100 elements, with each element corresponding to a bill. S = $\{\xi_1, \xi_2, \ldots, \xi_{100}\}$ where ξ_i represents the i^{th} bill. All bills are equiprobable

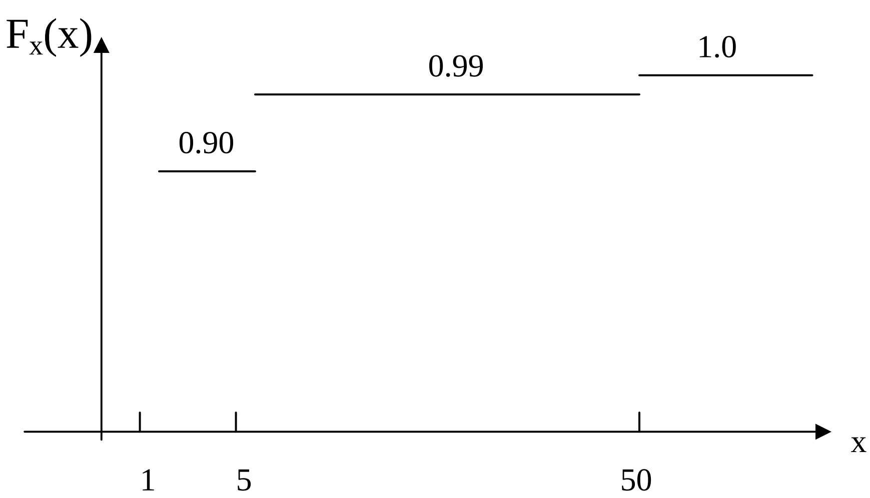
$$P[\{\xi_i\}] = 1/100$$

b) Describe sample space. Find Probabilities.

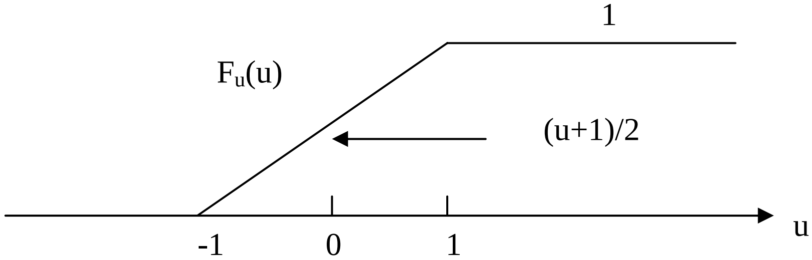
X is the denomination of a bill. There are three denominations, so: $S_x = \{1,5,50\}$. The probability of a denomination is proportional to the number of bills with that denomination:

$$P[X = 1] = P[\{\xi : X(\xi) = 1\}] = 90/100 = 0.90$$
$$P[X = 5] = P[\{\xi : X(\xi) = 5\}] = 9/100 = 0.09$$
$$P[X = 50] = P[\{\xi : X(\xi) = 50\}] = 1/100 = 0.01$$

#3.6 Plot $F_x(x)$ in problem #1



#3.12 Let U be uniform r.v. in the interval [-1,1].
Find P[U > 0], P[U < 5], P[|U| < 1/3], P[1/3 < U < 1/2], and P[|U| ≥ 3/4]



$P[U > 0] = 1 - P[U \leq 0] = 1 - F_u(0) = 1/2$

$P[U < 5] = 1$

$P[|U| < 1/3] = P[-1/3 < U < 1/3] = F_u(1/3) - F_u(-1/3) = 2/3 - 1/3 = 1/3$

$P[1/3 < U < 1/2] = F_u(1/2) - F_u(1/3) = 3/4 - 2/3 = 1/12$

$P[|U| \geq 3/4] = 1 - P[|U| < 3/4] = 1-[F_u(3/4) - F_u(-3/4)]=1-[7/8-1/8] =1/4$

#3.19

$$f_x(x) = \begin{cases} cx(1-x) & 0 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

a) find c? We use the fact that the pdf must integrate to one:

$$1 = \int_0^1 f_X(x)dx = c\int_0^1 x(1-x)dx = c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{c}{6} \quad \Rightarrow \quad c = 6$$

b) find P[1/2 ≤ X ≤ 3/4]?

$$P\left[\frac{1}{2} \leq X \leq \frac{3}{4}\right] = 6\int_{1/2}^{3/4} x(1-x)dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/2}^{3/4} = 0.34375$$

c) find $F_x(x)$? for $0 \leq x \leq 1$

$$F_X(x) = \int_0^x f_X(x')dx' = 3x^2 - 2x^3$$

for x < 0 , $F_x(x) = 0$; for x > 1, $F_x(x) = 1$,

#3.32 X is binomial r.v. n trials p = prob. of success

a) Let I_k denote the outcome of the kth Bernoulli trial. The probability that the single event occurred in the kth trial is:

$$P[I_k = 1 \mid X = 1] = \frac{P[I_k = 1 \quad and \quad I_j = 0 \quad for\ all\ \ j \neq k]}{P[X = 1]}$$

$$\overset{kth\ outcome}{} = \frac{P[0\ 0 ... 1\ \ 0...0]}{P[X = 1]}$$

$$= \frac{p(1-p)^{n-1}}{\binom{n}{1} p(1-p)^{n-1}} = \frac{1}{n}$$

Thus the single event is equally likely to have occurred in any of the n trials

b) Suppose X = 2. Find prob. two events occurred in j^{th} and k^{th} trials
j < k

The probability that the two successes occurred in trials j and k is:

$$P[I_j = 1, I_k = 1 \mid X = 2] = \frac{P[I_j = 1, I_k = 1, I_m = 0 \quad for\ all\ m \neq j, k]}{P[X = 2]}$$

$$= \frac{p^2(1-p)^{n-2}}{\binom{n}{2} p^2(1-p)^{n-2}} = \frac{1}{\binom{n}{2}}$$

Thus all $\binom{n}{2}$ *possible devices of j and k are equally likely.*

c)In what sense are successes distributed "completely at random".
If X = k then location of successes selected at random from among the

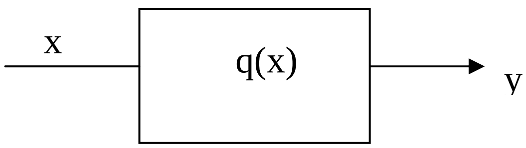
$\binom{n}{k}$ possible permutations.

#3.51 r.v. X has Laplacian pdf

$$f_X(x) = \frac{\alpha e^{-\alpha|x|}}{2} \qquad where \quad \alpha > 0, -\infty < x < \infty$$

X is input to 8-level quantizer (Ex: 3.19)

Find pmf. Find prob. X exceeds range ± 4d



Since symmetric pdf, we utilize it to find:

$$P[Y = 3.5d] = P[Y = -3.5d] = \int_{-\infty}^{-3d} \frac{\alpha e^{\alpha x}}{2} dx = \frac{1}{2} e^{-3\alpha d}$$

$$P[Y = 2.5d] = P[Y = -2.5d] = \int_{-3d}^{-2d} \frac{\alpha e^{\alpha x}}{2} dx = \frac{1}{2}\left(e^{-2\alpha d} - e^{-3\alpha d}\right)$$

$$P[Y = 1.5d] = P[Y = -1.5d] = \int_{-2d}^{-d} \frac{\alpha e^{\alpha x}}{2} dx = \frac{1}{2}\left(e^{-\alpha d} - e^{-2\alpha d}\right)$$

$$P[Y = 0.5d] = P[Y = -0.5d] = \int_{-d}^{0} \frac{\alpha e^{\alpha x}}{2} dx = \frac{1}{2}\left(1 - e^{-\alpha d}\right)$$

$$P[|Y| > 4d] = 2\int_{-\infty}^{-4d} \frac{\alpha e^{\alpha x}}{2} dx = e^{-4\alpha d}$$

#3.56 If current X is zero mean Gaussian r.v. Find pdf of power ($Y = RX^2$)

$X \sim N(0, \alpha^2)$

$$F_{power}(y) = P[RX^2 \le y] = P\left[-\sqrt{y/R} \le X \le \sqrt{y/R}\right]$$
$$= F_x\left(\sqrt{y/R}\right) - F_x\left(-\sqrt{y/R}\right) \qquad y \ge 0$$

$$f_{power}(y) = \frac{f_x\left(\sqrt{y/R}\right)}{2\sqrt{y/R}}\frac{1}{R} - \frac{f_x\left(-\sqrt{y/R}\right)}{-2\sqrt{y/R}}\frac{1}{R}$$

$$= \frac{f_x\left(\sqrt{y/R}\right)}{2R\sqrt{y/R}} + \frac{f_x\left(-\sqrt{y/R}\right)}{2R\sqrt{y/R}} = \frac{1}{\sqrt{2\pi\alpha^2 Ry}}\exp\left(-\frac{y}{2\alpha^2 R}\right)$$

#3.74 Let Y = Acos(wt) + C E[A] = m
w, C : constants $\sigma_A^2 = \sigma^2$

E[Y] = E[Acoswt + C] = E[Acoswt] + C = E[A]coswt + C = mcoswt + C

$\sigma_Y^2 = E[Y^2] - E[Y]^2$

$E[Y^2] = E[A^2\cos^2 wt + 2AC\cos wt + C^2] = E[A^2]\cos^2 wt + 2C\cos wt E[A] + C^2$
$= (\sigma^2 + m^2)\cos^2 wt + 2mC\cos wt + C^2$

$\sigma_Y^2 = E[Y^2] - E[Y]^2$
$= (\sigma^2 + m^2)\cos^2 wt + 2mC\cos wt + C^2 - m^2\cos^2 wt - 2mC\cos wt - C^2$
$= \sigma^2\cos^2 wt$

#3.88 Find characteristic function of the uniform r.v. in the interval [a,b]
Find mean and variance.

$$\Phi_X(w) = \int_{-\infty}^{\infty} f_X(x)e^{jwx}dx = \int_a^b \frac{1}{b-a}e^{jwx}dx = \frac{e^{jwb} - e^{jwa}}{jw(b-a)}$$

$$E[X] = \frac{1}{j}\frac{d\,\Phi_X(w)}{dw}\bigg|_{w=0} = -\frac{1}{b-a}\left[-\frac{1}{2}b^2 + \frac{1}{2}a^2\right] = \frac{1}{2}(b+a)$$

$$E[X^2] = \frac{1}{j^2}\frac{d^2\,\Phi_X(w)}{dw^2}\bigg|_{w=0} = -\frac{1}{j(b-a)}\left[-\frac{1}{3}jb^3 + \frac{1}{3}ja^3\right] = \frac{1}{3}(b^2 + ab + a^2)$$

$$VAR[X] = E[X^2] - E[X]^2 = \frac{1}{3}(b^2 + ab + a^2) - \frac{1}{4}(b+a)^2 = \frac{1}{12}(b-a)^2$$