Chapter 2 – Basics of Probability Theory

Random Experiments

A random experiment is specified by stating:

a) an experimental procedure, and
b) a set of one or more measurements (observation).

Experiments in Ex 2.1

EXAMPLE 2.2 The sample spaces corresponding to the experiments in Example 2.1 are given below using set notation:

$S_1 = \{1, 2, \ldots, 50\}$

$S_2 = \{(1, b), (2, b), (3, w), (4, w)\}$

$S_3 = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}, \text{TTH}, \text{THT}, \text{HTT}, \text{TTT}\}$

$S_4 = \{0, 1, 2, 3\}$

$S_5 = \{0, 1, 2, \ldots, N\}$

$S_6 = \{1, 2, 3, \ldots\}$

$S_7 = \{x : 0 \le x \le 1\} = [0, 1]$ See Fig. 2.1(a).

$S_8 = \{t : t \ge 0\} = [0, \infty)$

$S_9 = \{t : t \ge 0\} = [0, \infty)$ See Fig. 2.1(b).

$S_{10} = \{v : -\infty < v < \infty\} = (-\infty, \infty)$

$S_{11} = \{(v_1, v_2) : -\infty < v_1 < \infty \text{ and } -\infty < v_2 < \infty\}$

$S_{12} = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ See Fig. 2.1(c).

$S_{13} = \{(x, y) : 0 \le y \le x \le 1\}$ See Fig. 2.1(d).

S_{14} = set of functions $X(t)$ for which $X(t) = 1$ for $0 \le t < t_0$ and $X(t) = 0$ for $t \ge t_0$, where $t_0 > 0$ is the time when the component fails.

Observations:

- Same procedure but different observations: E_3, E_4
- Multiple observations: E_2, E_3, E_{11}, E_{12}, E_{13}
- Sequential experiments: E_3, E_4, E_5, E_6, E_{12}, E_{13}

Sample Space

Outcome ≡ Sample point: A result of an experiment that cannot be decomposed into other results.

- Outcomes are mutually exclusive
- Sample Space ≡ S = Set of all possible outcomes.

Example 2.2

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$S_4 = \{0, 1, 2, 3\}$

$S_5 = \{0, 1, 2, \ldots, N\}$

$S_6 = \{1, 2, 3, \ldots\}$

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$S_{11} = \{(v_1, v_2) : -\infty < v_1 < \infty \text{ and } -\infty < v_2 < \infty\}$

$S_{12} = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ See Fig. 2.1(c).

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S_{14} = set of functions $X(t)$ for which $X(t) = 1$ for $0 \le t < t_0$ and $X(t) = 0$ for $t \ge t_0$, where $t_0 > 0$ is the time when the component fails.

Observations:

- Discrete Sample Space: S is countably finite or infinite.
  - Ex: $S_1 - S_5$: countably finite
  - S_6 : countably infinite
- Continuous Sample Space: S is not countable.
  - Ex: $S_7 - S_{13}$: continuous S
- Multi-dimensional Sample Space: One or more observations from an experiment.
  - Ex: S_2, S_{11}, S_{12}, S_{13}, and S_3
- Multi-dimensional sample spaces can be written as Cartesian product of other sets.
  - Ex: $S_{11} = R\text{x}R$

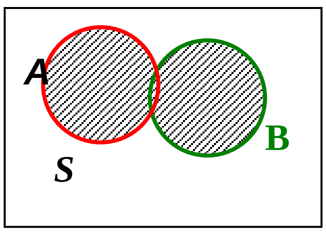
Events: Set of outcomes from S that satisfy certain conditions.

- Certain Event = S; always occurs.
- Null Event = $\varnothing$; never occurs.
  (See Ex: 2.3 pp 26-27 for events.)
- Elementary Event: An event from an S that contains a Single outcome

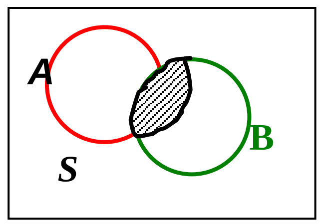
Ex: A_2 and A_7 in Ex 2.3

Set Operations & Venn Diagrams

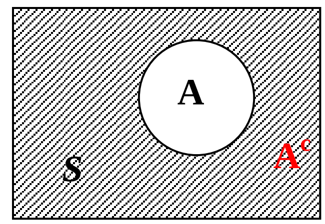
Union: $A \cup B$: Set of outcomes either in A, or in B, or in both.
Intersecton: $A \cap B$: Set of outcomes that are in both A and B.
Complement: A^c : Set of outcomes that are <u>not</u> in A.



$A \cup B$

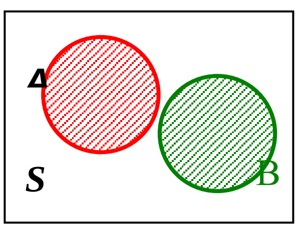


$A \cap B$

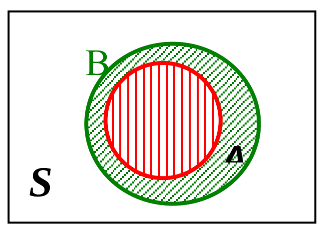


A^c

Mutually Exclusive ≡ Disjoint : if $A \cap B = \phi$; disjoint events cannot occur simultaneously.



$A \cap B = \varnothing$



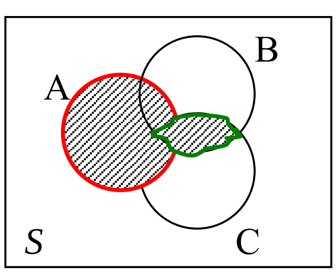
$A \subset B$

Implication : If an event A is a subset of an event B, then A implies B: $A \subset B$
Equal : A and B are equal if they contain same outcomes.

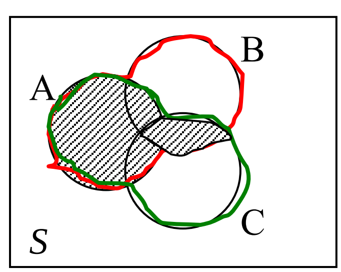
Properties:

1) **Commutativity:** $A \cup B = B \cup A$ and $A \cap B = B \cap A$
2) **Associativity:** $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
3) **Distributivity:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4) **DeMorgan's Rule:** $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

Ex : 1 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

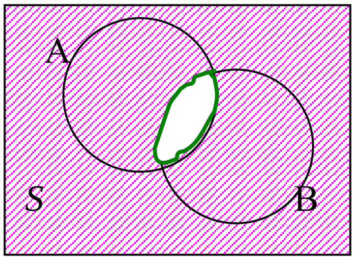


$A \cup (B \cap C)$

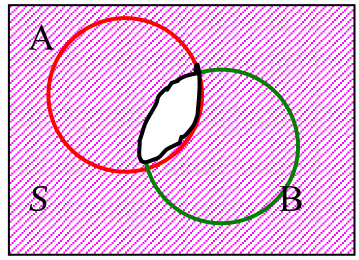


$(A \cup B) \cap (A \cup C)$

Ex : 2 $(A \cap B)^c = A^c \cup B^c$

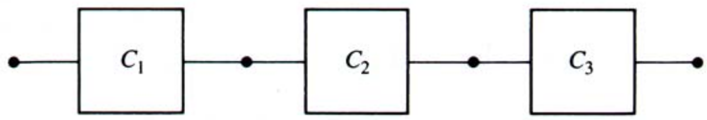


$(A \cap B)^c$



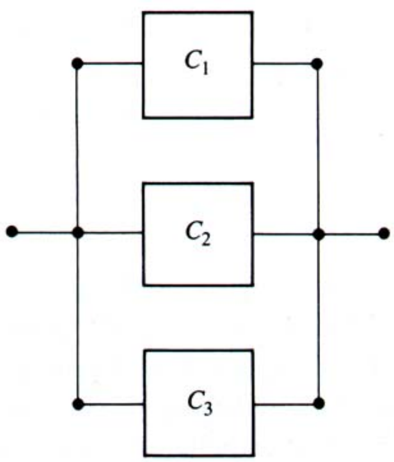
$A^c \cup B^c$

Ex: 2.5 System Reliability: Assume A_k : Event "Component C_k is functioning."

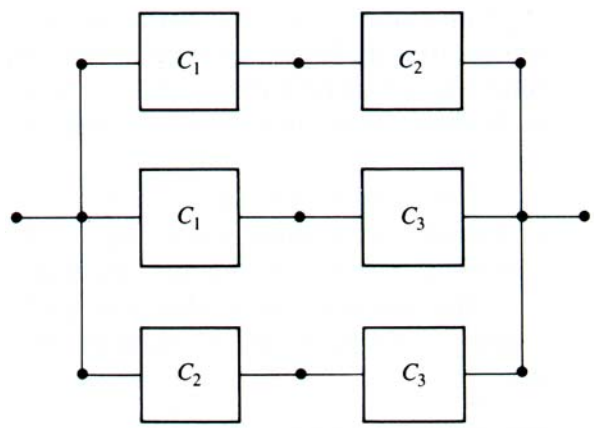


(a) Series system

$D_a = A_1 \cap A_2 \cap A_3$



(b) Parallel system



(c) Two-out-of-three system

$D_b = A_1 \cup A_2 \cup A_3$

$$D_c = (A_1 \cap A_2 \cap A_3) \cup (A_3^c \cap A_1 \cap A_2) \cup (A_2^c \cap A_1 \cap A_3) \cup (A_1^c \cap A_2 \cap A_3)$$

Multiple Unions and intersections:

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \ldots \cup A_n \quad and \quad \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \ldots \cap A_n$$

Kolmogorov's Axioms of Probability and Corollaries

Let E be a random experiment with sample space S . A probability law for E is a rule that assigns to each event A a number P(A) satisfying:

I. $0 \le P(A)$

II. $P(S) = 1$

III. If $A \cap B = \varnothing$, then $P(A \cup B) = P(A) + P(B)$

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P(A_k)$$

III' If $A \bigcap B \neq \phi$ then $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$

Corollary 1: $P(A^c) = 1 - P(A)$

Proof:

Since $A \cap A^c = \varnothing$, we have $P(A \cup A^c) = P(A) + P(A^c)$

Since $S = A \cup A^c$ for II, $1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$

$$\Rightarrow \boldsymbol{P(A^c) = 1 - P(A)} \quad \textbf{Q.E.D}$$

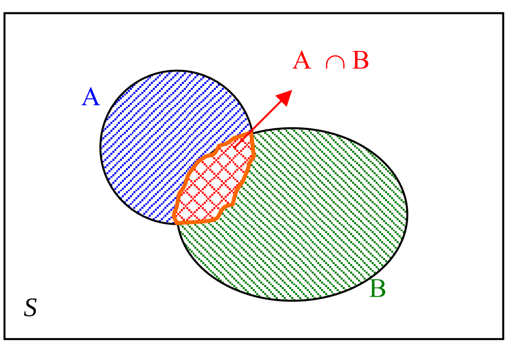
Corollary 2: $P(A) \le 1$

Corollary 2: $P(\varnothing) = 0$

Corollary 4: If $A_1, A_2, \ldots A_n$ are pairwise disjoint, then

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P(A_k) \qquad \text{for } n \ge 2$$

Corollary 5: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$



Proof: $P(A \cup B) = P(A \cap B^c) + P(B \cap A^c) + P(A \cap B)$

$P(A) = P(A \cap B^c) + P(A \cap B)$

$P(B) = P(B \cap A^c) + P(A \cap B)$, then

$$P(A \cup B) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B)$$
$$= P(A) + P(B) - P(A \cap B)$$

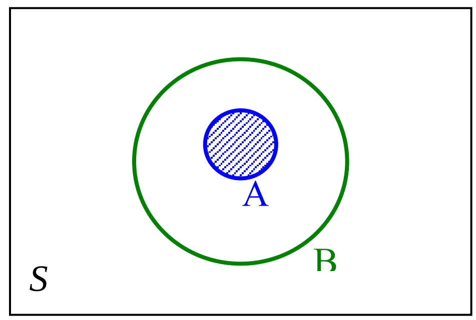
Q.E.D.

Corollary 6:

$$P[\bigcup_{k=1}^{n} A_k] = \sum_{j=1}^{n} P(A_j) - \sum_{j<k}^{n} P(A_j \cap A_k) + \cdots$$
$$+ (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

Corrollary 7: If $A \subset B$, then $P(A) \le P(B)$

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\text{Since } \ge 0} \ge P(A)$$



Discrete vs. Continuous Samples

Discrete Sample Spaces

- Given a finite sample space $S = \{a_1, a_2, \ldots, a_n\}$ and all distinct elements are disjoint, then the probability of an event
  $B = \{a_1', a_2', \ldots, a_m'\} \Rightarrow P(B) = P\{a_1', a_2', \ldots, a_m'\} = P(a_1') + \ldots P(a_m')$,
  and if an event is $D = \{b_1', b_2', \ldots\} \Rightarrow P(D) = P(b_1') + P(b_2')\ldots$

- Equally likely outcomes: Then the probability of an event is equal to the number of outcomes in the event divided by the total number of outcomes in S .

Ex: 2.7 Tossing: 3 coins

$\Rightarrow S_3 = \{HHH, \ldots., TTT\}$ and $P\{Ai\} = 1/8$

B = "Two heads in 3 tosses" = {HHT, HTH, THH}

$\Rightarrow P(B) = 3/8$

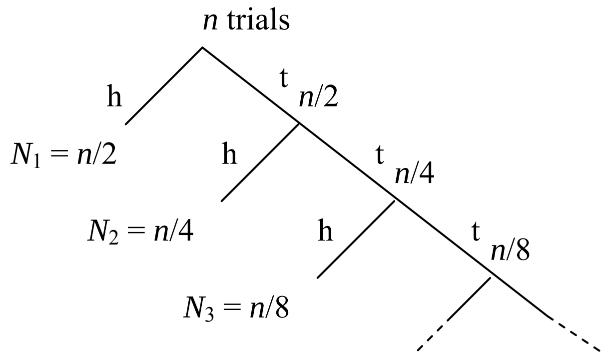
$S_4 = \{0,1,2,3\}$ occurrence of # of heads in 3 tosses

C : "Two heads in 3 tosses" : Assume S_4 has equally likely outcomes, then

$\Rightarrow P(C) = \underline{1/4}$???

Conclusion: Assumption is NOT true.

Ex: 2.8   Continue tossing until the FIRST heads shows up. Prob. Law?
$S = \{1,2,3,\ldots\ldots\}$



$$f_j = \frac{N_j}{n} = \left(\frac{1}{2}\right)^j \qquad j = 1,2,\ldots$$

A_j = "j tosses till first heads"

$$P(A_j) = \left(\frac{1}{2}\right)^j \qquad j = 1,2,\ldots$$

$$\sum_{j=1}^{\infty} P(A_j) = \sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j \quad but \quad \sum_{j=1}^{\infty} a^j \quad here \quad a = \frac{1}{2}$$

$$P(A) = \frac{1/2}{1-1/2} = 1$$

Conclusion. Eventually "heads" will show up!!!

Continuous Sample spaces: Real lines or regions in a plane.

- Prob. Laws in experiments with such spaces specify a rule for assigning numbers to intervals of a real line or a rectangular region in a plane.

Ex: 2.9   Pick x at random between "0" and "1".

P(b,a) = b-a   for $0 \le a \le b \le 1$
P(0, 0.5) = 0.5 - 0 = 0.5
P(0.5, 1) = 1 – 0.5 = 0.5

Let A: outcome in at least 0.3 away from center. Find P(A) = ?
P(A) = P[(0, 0.2) $\cup$ (0.8, 1)] = P(0, 0.2) + P(0.8, 1) = 0.4

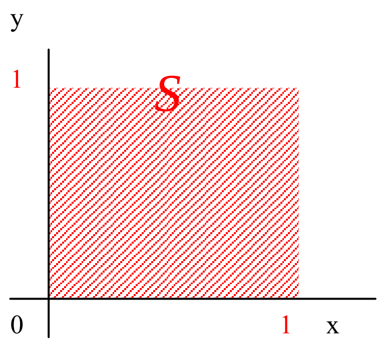
Ex: 2.10   Given: Proportion of chips whose lifetime exceeds t decreases exponentially at a rate α. Find prob. Law

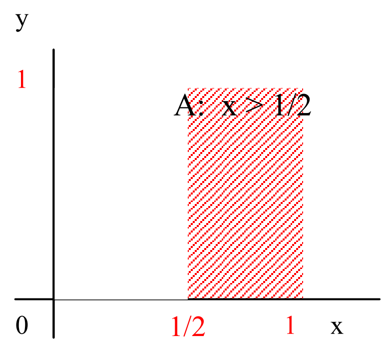
Solution: $S = (0, \infty)$
A = " t, ∞ " lifetime > t for t > 0
P(A) = $e^{-\alpha t}$ for t > 0 and $\alpha > 0$
$P(S) = P(0,\infty) = e^{-\alpha t}|_{t\to 0} = e^{-0} = 1$
P(r, u) = ?   Notice that: $(r,u) = (r,\infty) \cup (u,\infty)$

$$P(r,\infty) = P(r,u) + P(u,\infty)$$

$$P(r,u) = P(r,\infty) - P(u,\infty) = e^{-\alpha r} - e^{-\alpha u}$$

Ex: 2.11 E_{12} : two numbers x and y at random between 0 and 1.

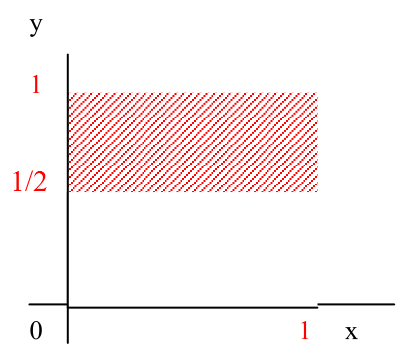


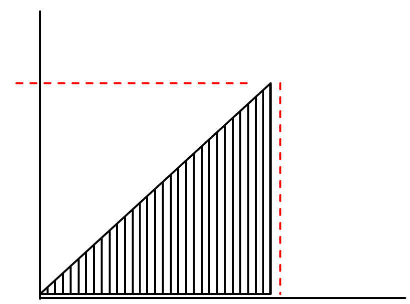


x > 1/2 but 0 ≤ y ≤ 1

then

P(A) = 1/2





B = y > ½
P(B) = 1/2

C = x > y
P(C) = 1/2

Counting Methods:

- The number of distinct ordered k-tuples $(x_1, x_2, \ldots, x_k)$ with components x_i from a set with n_i elements is # of k-tuples = $n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

1. Sampling with Replacement and with Ordering:

- "k objects from a set A having n distinct objects with replacement"

  $(x_1, x_2, \ldots, x_k)$ where $x_i \ \varepsilon$ A for i = 1, 2,…, k
  but $n_1 = n_2 = \ldots = n_k = n$
  $\therefore$ # of k-tuples = n^k (ordered)

Ex: 2.12 Five Balls 1 – 5. Two balls are selected with replacement. How many distinct ordered pairs? Prob. Two draws yield same number?

Probability two draws yield same # of 2-tuples = $5^2 = 25$

Assume all draws are equally probable, then
$P(A) = 5/25 = 1/5$

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)

(a) Ordered pairs for sampling with replacement.

	(1, 2)	(1, 3)	(1, 4)	(1, 5)
(2, 1)		(2, 3)	(2, 4)	(2, 5)
(3, 1)	(3, 2)		(3, 4)	(3, 5)
(4, 1)	(4, 2)	(4, 3)		(4, 5)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	

(b) Ordered pairs for sampling without replacement.

(1, 2)	(1, 3)	(1, 4)	(1, 5)
	(2, 3)	(2, 4)	(2, 5)
		(3, 4)	(3, 5)
			(4, 5)

(c) Pairs for sampling without replacement or ordering.

2. Sampling without Replacement and with Ordering:

- Given a population A of n distinct objects. Choose k-objects in succession without repl. $k \leq n$

  \# of possible outcomes: $n_1 = n$; $n_2 = n-1$; … ; $n_k = n-(k-1)$
  \# of distinct ordered k-tuples = $n(n-1)\ldots(n-k+1)$

Ex: 2.13 How many distinct ordered pairs?

$\Rightarrow$ (5)(4) = 20
Prob. (1st ball > 2nd ball) ? $\Rightarrow$ 10/20 = 0.5

Permutations of n-distinct objects:
(No replacement) Drawing objects until the urn is empty.
\# of permutations of n objects = $n(n-1)\ldots(2)(1) \equiv n!$

Stirling's Approximation to n-Factorial: For large n, the following is valid:

$$n! \sim \sqrt{2\pi}\,.n^{n+\frac{1}{2}}.e^{-n}$$

Ex: 2.15 # of permutations of 3-distinct objects {1,2,3}?

$3! = (3)(2)(1) = 6 \Rightarrow$ 123 312 231 132 213 321

Ex: 2.16 12 balls into 12 cells, more than 1 ball is O.K. to go to a cell. What is Prob. that all cells are occupied?

$n^{12} = 12^{12}$ possible placements of 12 balls in 12 cells!!!
# of placements that occupy 12 cells = n! = 12!
$\therefore$ Prob. that all cells are occupied is

$$\frac{12!}{12^{12}} = \frac{12}{12}\,\frac{11}{12}\cdots\frac{1}{12} = 5.37x10^{-5}$$

3. Sampling without Replacement and without ordering:

- Pick k-objects from a set of n-distinct objects without replacement or ordering
  $\Rightarrow$ Combination of size k.

Let C_k^n be the # of combinations of size k from a population n.

$C_k^n k!$ = total # of distinct ordered samples of k objects
= $n(n-1)\ldots(n-k+1)$

k! possible orders in which k objects could be selected.

The number of different combinations of size k from population n is

$$C_k^n = \frac{n!}{k!(n-k)!} \equiv \binom{n}{k} \qquad \textit{binomial coefficient}$$

Ex: 2.17 Two objects from A = {1,2,3,4,5}

$$C_2^5 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5*4*3*2*1}{(2*1)(3*2*1)} = 10 \quad \textit{pairs}$$

(See Figure 2.10c)

Ex: 2.19 50 items containing 10 defective elements. Set of 10 items tested

P(A) ≡ Prob. that exactly 5 items tested are defective?

$$C_{10}^{50} = \binom{50}{10} = \frac{50!}{10!*40!}$$

N_1 = # of ways of selecting 5 items from a set of 10 def.

N_2 = # of ways of selecting 5 items from a set of 40 good items

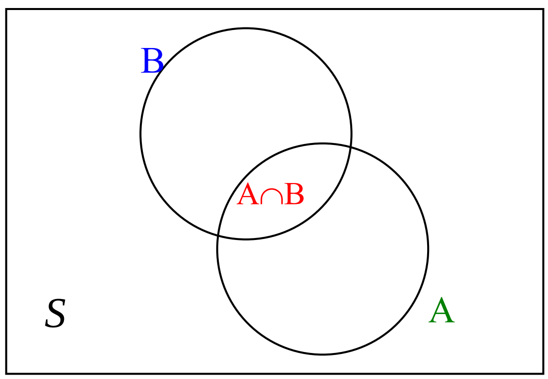
$\Rightarrow N_1 N_2$ = # of ways of selecting 5 def + 5 good items.

$$P(A) = \frac{N_1 N_2}{\binom{50}{10}} = \frac{\binom{10}{5}\binom{40}{5}}{\binom{50}{10}} = \frac{\frac{10!}{5!*5!}\frac{40!}{5!*35!}}{\frac{50!}{40!*10!}} = \frac{10!*40!*40!*10!}{5!*5!*5!*35!*50!} = 0.016$$

Conditional Probability: Knowing that B has occurred:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{for } P(B) > 0$$

Relative frequency of A among those satisfying the conditions of B at the same time.



Ex: 2.21 Two black balls (1,2) Two white balls (3,4)

S = {(1,b), (2,b), (3,w), (4,w)}

A = Black_ball selected = {(1,b), (2,b)}

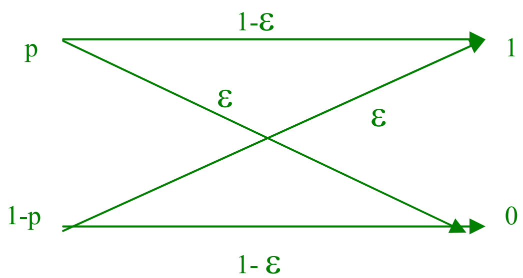
B = Even_ball selected = {(2,b), (4,w)}

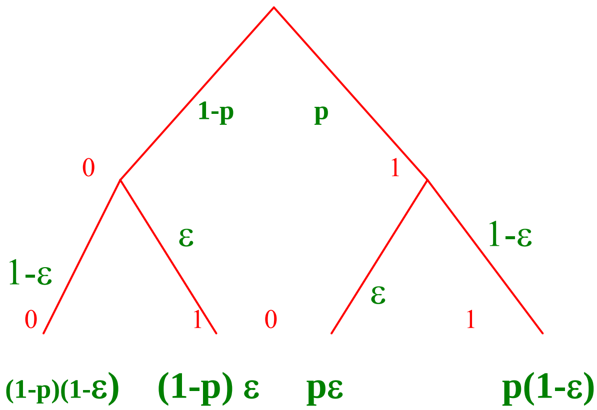
C = Ball number > 2 = {(3,w), (4,w)}

$$P(A|B) = ? \qquad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 0.5$$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0}{0.5} = 0$$

Ex: Binary Symmetric Channel:





A_i = Event "input was i"

B_i = event "decision was i"

Find $P(A_i \cap B_j)$ for $i = 0,1;\ j = 0,1$

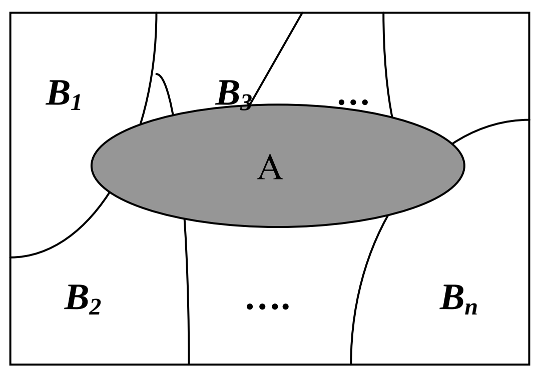
$P(A_0 \cap B_0) = (1\text{-}p)(1\text{-}\varepsilon)$
$P(A_0 \cap B_1) = (1\text{-}p)(\varepsilon)$
$P(A_1 \cap B_0) = p\varepsilon$
$P(A_1 \cap B_1) = p(1\text{-}\varepsilon)$

Disjoint partition of *S* into *n* events

$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n)$

$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \ldots + P(A/B_n)P(B_n)$

TOTAL PROBABILITY THEOREM



Ex: 2.25 Exponential law for memory chip failure

Given: Rate of failure = 1000α

Good chip = 1-p; bad chips = p

P(C)? where C: Randomly selected chip is functioning after t seconds

G: Chip is good B: Chip is bad.

$$P(C) = P(C/G)P(G) + P(C/B)P(B) = P(C/G)(1-p) + P(C/B)p$$
$$= (1-p)e^{-\alpha t} + pe^{-1000\alpha t}$$

Bayes Theorem: Let $B_1, B_2, \ldots, B_n$ be a partition of *S* and A occurs. What is the probability of event B_j?

$$P(B_j/A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A/B_j)P(B_j)}{\sum_{k=1}^{n} P(A/B_k)P(B_k)}$$

Ex: 2.26 Assume input is equally likely in BSC example.
Find which input is more probable
Given that receiver has output a "1"
A_k: Event that input was "k" for k = 0,1
B_1: Event that receiver output was a "1"

$$P(B_1) = P(B_1/A_0) * P(A_0) + P(B_1/A_1) * P(A_1) = \varepsilon * \frac{1}{2} + (1-\varepsilon) * \frac{1}{2} = \frac{1}{2}$$

$$P(A_0/B_1) = \frac{P(B_1/A_0) * P(A_0)}{P(B_1)} = \frac{\varepsilon/2}{1/2} = \varepsilon$$

If $\varepsilon < ½$ then the input "1" is more likely when a "1" is observed at the output of the channel.

$$P(A_1/B_1) = \frac{P(B_1/A_1) * P(A_1)}{P(B_1)} = \frac{(1-\varepsilon)/2}{1/2} = 1-\varepsilon$$

Independent Events & Probabilities: If occurrence of an event B does not change the prob. of another event A, then events A and B are independent of each other.

A & B are independent if we have:

$$P(A) = P(A/B) = \frac{P(A \cap B)}{P(B)}$$

or equivalently,

$$P(A \cap B) = P(A)P(B)$$

Ex: 2.28 A: {(1,b),(2,b)} "black ball selected"
B: {(2,b),(4,w)} "even-numbered ball selected"
C: {(3,w),(4,w)} "number of ball is greater than 2"
Are A & B; A & C indep?
P(A) = P(B) = P(C) = 1/2
$P(A \cap B)$ = P[(2,b)] = 1/4 P(A)P(B) = (1/2)(1/2) = 1/4

So the knowledge of the occurrence of B does not change the probability of the occurrence of A since

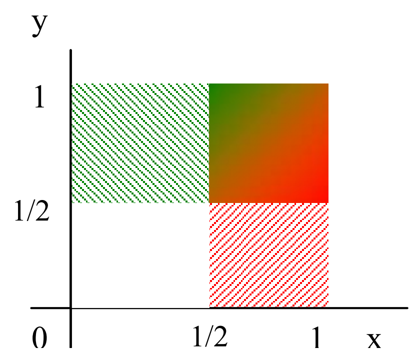
$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2 \qquad \text{and} \qquad P(A) = 1/2$$

But: $A \cap C = 0 \quad \Rightarrow \quad P(A \cap C) = 0$ and $P(A)P(C) = (1/2)(1/2) = 1/4$
Therefore, A and C are NOT independent.

Ex: 2.29 Given: $A = \{x > 0.5\}$; $B = \{y > 0.5\}$; $C = \{x > y\}$
A & B and A & C independent?

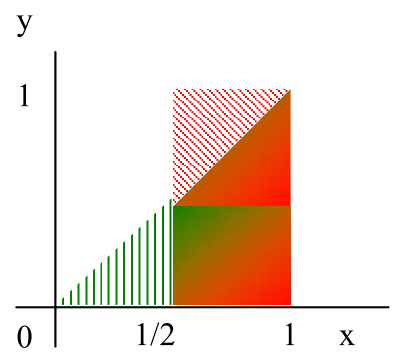
$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2 = P(A)$$

A & B are independent.



However,

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{3/8}{1/2} = 3/4$$



But $P(A) = 1/2 \neq 3/4$
$\therefore$ A and C are not independent events

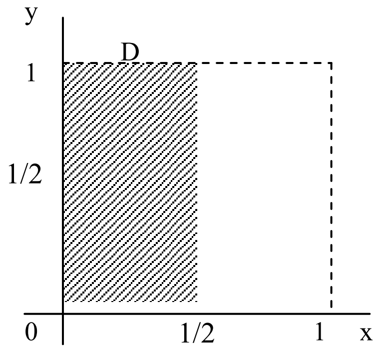
$\Rightarrow$ knowing that $x > y$ increases the probability that $x > 0.5$

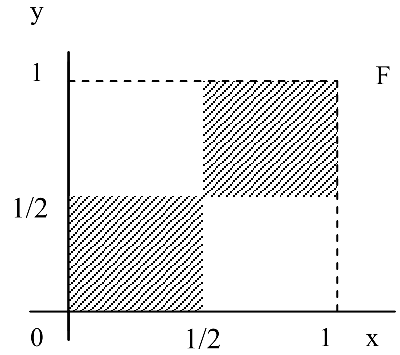
Three Independent Events:

A, B and C are independent if the probability of intersection of any pair or triplet of events is equal to the product of $P(A)P(B)P(C)$

$\Rightarrow P(A \cap B \cap C) = P(A \cap B)P(C) = P(A)P(B)P(C)$
$\Rightarrow P(A \cap B) = P(A)P(B)$ $P(A \cap C) = P(A)P(C)$
$\Rightarrow P(B \cap C) = P(B)P(C)$

Ex: 2.30 Given:
$B = \{y > 1/2\}$;
$D = \{x < 1/2\}$;
$F = \{(x < 1/2 \text{ and } y < 1/2) \cup (x > 1/2 \text{ and } y > 1/2\}$





$$P(B \cap D) = 1/4 \quad Also \quad P(B).P(D) = 1/2 * 1/2 = 1/4$$
$$P(B \cap F) = 1/4 \quad Also \quad P(B).P(F) = 1/2 * 1/2 = 1/4$$
$$P(D \cap F) = 1/4 \quad Also \quad P(D).P(F) = 1/2 * 1/2 = 1/4$$

But,
$B \cap D \cap F = \varnothing \quad \Rightarrow \quad P(B \cap D \cap F) = 0$
and
$P(B)P(D)P(F) = (1/2)(1.2)(1/2) = 1/8$

Three events are not independent. Even though they are pairwise independent.

Ex: 2.32
System = A controller (A) and 3 peripherals (B_1, B_2, B_3)
System is UP if A is functioning + at least 2 peripherals are ON
A : Controller in ON
F : Two or more periph are ON
$= (B_1 \cap B_2 \cap B_3) \cup (B_1 \cap B_2 \cap B_3^C) \cup (B_1 \cap B_3 \cap B_2^C) \cup (B_2 \cap B_3 \cap B_1^C)$

Note each peripheral is disjoint from others and
let $P(B_i) = 1\text{-}a$ and
$P(B_i^C) = a$;
$P(A^C) = p$ and
$P(A) = 1\text{-}p$
$P(F) = P(B_1)P(B_2)P(B_3) + P(B_1)P(B_2)P(B_3^C) + P(B_1)P(B_3P(B_2^C)$
$+ \ P(B_2)P(B_3)P(B_1^C)$
$= (1\text{-}a)^3 + 3(1\text{-}a)^2a$

System is UP : $P(\text{“UP”}) = P(A \cap F) = P(A)P(F) = (1\text{-}p)[\ (1\text{-}a)^3 + 3(1\text{-}a)^2a]$
Assume a particular set of values:
a = 10%

$p = 20\%$

$\Rightarrow$ P("UP") = 77.8%

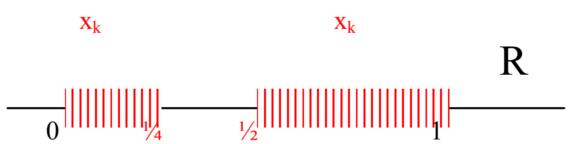
Mostly due to controller failures of 20%.
Since $P(F) = (0.9)^3 + 3(0.9)^2(0.1) = 97.2\%$

Sequential Experiments:

Sequences of Independent Experiments: They are set of simpler sub-experiments: $E_1, E_2, \ldots E_n$.
The outcome $s = (s_1, s_2, \ldots , s_n)$ from the Cartesian product space S of $S = S_1 x S_2 x \ldots x S_n$.
If $A_1, A_2, \ldots, A_n$ are the events such that A_k concerns only of the sub-experiment E_k and if the sub-experiments are independent, then

$$P(A_1 \cap A_2 \ldots \cap A_n) = P(A_1)P(A_2)\ldots P(A_n)$$

Ex: 2.33 10 numbers are selected from [0, 1].
Find the probability that first 5 numbers < 1/4 and the last 5 > ½.
Let x_k be sequence of 10 numbers



$A_k = \{x_k < ¼\}$ for $k = 1,2, \ldots, 5 \rightarrow P(A_k) = 1/4$
$A_k = \{x_k > ½\}$ for $k = 6,7, \ldots, 10 \rightarrow P(A_k) = 1/2$

Since each number is drawn independent of the previous draw we have:
$P(A_1 \cap \ldots \cap A_{10}) = P(A_1)P(A_2)\ldots P(A_{10}) = (1/4)^5(1/2)^5 = 3.05 \text{ x } 10\text{-}5$

Bernoulli Trial:

Perform an experiment once and note whether a particular event A occurs or not. (Binary outcome)

Typical question: What is the probability of k occurrences in n independent trials?

Binomial Prob. Law:

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad for \quad k = 0,1,\ldots,n$$

Ex: 2.34 Toss a coin $n = 3$ times with probabilities: $P(H) = p$ $P(T) = 1-p$

$P(HHH) = P(H)P(H)P(H) = p^3$
$P(HHT) = p^2(1-p) = P(HTH) = P(THH)$
$P(HTT) = p(1-p)^2 = P(THT) = P(TTH)$
$P(TTT) = (1-p)^3$

Let k be the number heads in 3 trials, then:

$$\left.\begin{array}{l} P(k=0) = (1-p)^3 \;;\; P(k=1) = 3p(1-p)^2 \\ P(k=2) = p^2(1-p) \;;\; P(k=3) = p^3 \end{array}\right\} \text{From above}$$

Alternatively, from Binomial Theorem:

$$P_3(0) = \frac{3!}{3!0!} p^0 (1-p)^3 = \frac{6}{6\cdot 1} 1\cdot(1-p)^3 = (1-p)^3 = P(k=0)$$

$$P_3(1) = \frac{3!}{2!1!} p^1 (1-p)^2 = \frac{6}{2} p^1 \cdot (1-p)^2 = 3p(1-p)^2 = P(k=1)$$

$$P_3(3) = \frac{3!}{3!0!} p^3 = p^3 = P(k=3)$$

$$P_3(2) = \frac{3!}{1!2!} p^2 (1-p)^1 = \frac{6}{2} p^2 \cdot (1-p)^1 = 3p^2(1-p) = P(k=2)$$

Iterative Formula for Binomial Prob. Law:

Given $P_n(k)$ is known, then

$$P_n(k+1) = P_n(k)\frac{(n-k)p}{(k+1)(1-p)}$$

Ex: 2.36 k: # of non-silent speakers in a group of 8.

$P(k^{th}$ speaker is active$) = 1/3$
$P(k > 6) = ?$

$$P(k>6) = P(k=7) + P(k=8) = \binom{8}{7}(1/3)^7(2/3) + \binom{8}{8}(1/3)^8$$

$$P(k=7) = \frac{8!}{7!\cdot 1!}(1/3)^7(2/3) = \frac{8}{1}(1/3)^7(2/3) = (16)(1/3)^8 = 0.002438$$

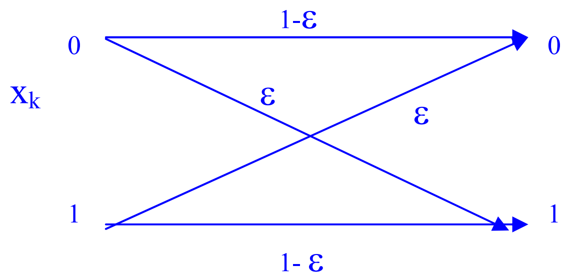
$$P(k=8) = P(k=7)\frac{(8-7)(1/3)}{(8)(2/3)} = 0.00152$$

$$P(k>6) = 2 * 0.00152 = 0.00259$$

Only (1/4)% of the time more than 6 speakers are talking simultaneously!!!

Ex: 2.37 Error Correction Coding in Binary Symmetric Channel

Input Output $\varepsilon = 10^{-3}$



x_k = 3 bits

y_k : majority of 3 digits

Receiver makes wrong decision if "2 or more" errors occur in the channel.

$$Pe = P(k \geq 2) = P(k=2) + P(k=3) = \binom{3}{2} 0.001^2 (0.999) + \binom{3}{3} 0.001^3 \approx 3x10^{-6}$$

Reliability of system increased by ~ 300 times at a rate reduction to 1/3

Multinomial Probability Law:

Let $B_1, B_2, \ldots, B_m$ be a partition of *S* with $P(B_j) = p_j$

Since the events are mutually exclusive:

$p_1 + p_2 + \ldots + p_m = 1$

Suppose n indep. Trials are done and k_j be # of times B_j occurs. The probability of the vector $(k_1, \ldots, k_m)$ satisfies:

$$P(k_1, k_2, \ldots, k_m) = \frac{n!}{k_1! k_2! \cdots k_m!} p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$$

Ex: 2.39 Pick 10 telephone numbers from a directory and note the last digit. A = Prob. that each digit 0 to 9 obtained once?

M = 10 digits

$P(k_i) = 1/10$ equally likely

n = 10 trials

$$P(A) = \frac{10!}{1! \cdot 1! \cdots 1!} (0.1)^1 (0.1)^1 \cdots (0.1)^1 = (10!)(1/10)^{10} \approx 3.6x10^{-4}$$

Very Small !!!

Geometric Prob. Law:

Repeat Bernoulli tirals until the occurrence of 1st success and stop.

m = # of trials until stop.

P(m) : Prob. that m trials are needed since the first m-1 trials resulted in failures.

$$P(m) = P[A_1^C A_2^C \ldots A_{m-1}^C A_m]$$

If $P(A_i) = p$ then $P(A_i^C) = 1-p$ and $P(m) = (1-p)^{m-1} p$ for m = 1, 2,…

Note: 1)

$$\sum_{m=1}^{\infty} P(m) = 1?$$

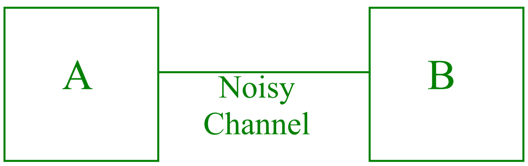
Let q = 1-p

$$\sum_{m=1}^{\infty} P(m) = \sum_{m=1}^{\infty} p(1-p)^{m-1} = \sum_{m=1}^{\infty} pq^{m-1} = p\sum_{m=1}^{\infty} q^{m-1} = p\frac{1}{1-q} = p\frac{1}{p} = 1$$

Note: 2) Probability that more than ***k*** trials needed before stopping:

$$P(m>k) = p\sum_{m=k+1}^{\infty} q^{m-1} = pq^k \sum_{j=0}^{\infty} q^j = pq^k \frac{1}{1-q} = q^k$$

Ex: 2.40 Error Control via Retransmission:



If B detects an error it requests A to retransmit with Pr = 0.1
If E: Message to be transmitted > 2
Prob. P(E) = ?

P = 1-q = 0.9
P(E) = P(k > 2) = $q^2 = (0.1)^2 = 10^{-2}$
One out of 100 messages will be transmitted more than twice from A to B

Trials with Dependent Experiments (Trellis Diagrams):
Chain of experiments in which future of trials depends on the outcome of a current trial.

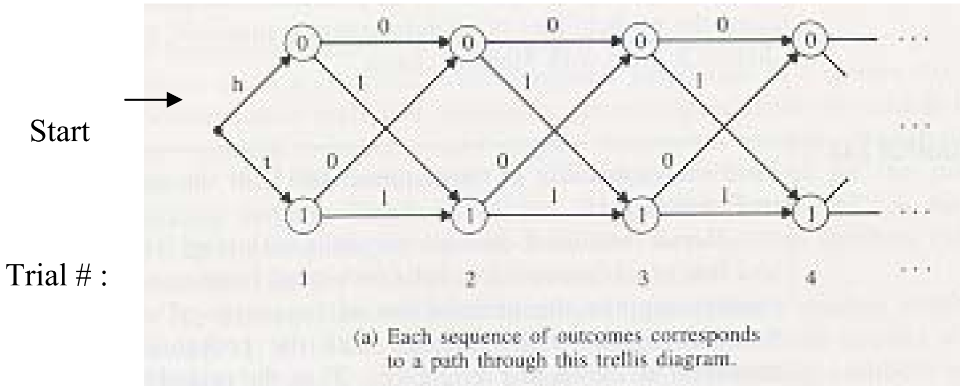
Ex: 2.41

Urn: 0 { one ball with label "1" → 1/3; two balls with label "0" → 2/3 }

Urn: 1 { one ball with label "0" → 1/6; five balls with label "1" → 5/6 }

Experiment:
1.) Flip a coin: if heads use urn 0; if tails use urn 1
2.) If urn 0 were the choice: Pick a ball from urn 0;
Otherwise from urn 1.
3._If the outcome has label "0": Stay with urn 0;
Otherwise stay with urn 1.

Trellis Diagram



(a) Each sequence of outcomes corresponds to a path through this trellis diagram.

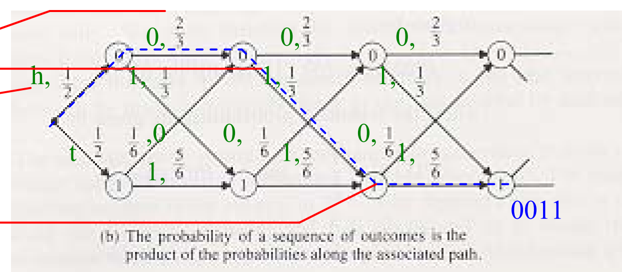
If the probability on S_n depends only on S_{n-1} (the most recent outcome) then these are called MARKOV CHAINS and

$$P(S_0, S_1, \ldots, S_n) = P(S_n \mid S_{n-1})P(S_{n-1} \mid S_{n-2})\cdots P(S_1 \mid S_0)P(S_0)$$

Ex: 2.42 **P(0011) = ?**

$P(0011) = P(1 \mid 1)P(1 \mid 0)P(0 \mid 0)\, P(0)$

$= (5/6)(1/3)(2/3)(1/2) = 5/54$



(b) The probability of a sequence of outcomes is the product of the probabilities along the associated path.

#2.3 Two dice tossed A = "Sum of face values"

a) S = {2, 3, …, 11, 12}

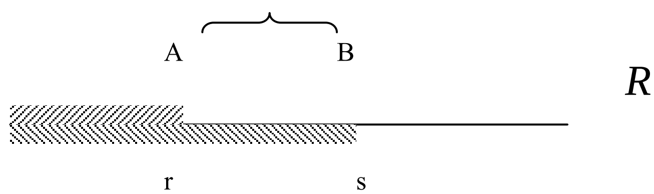
b) B = "Even Subset of {A}"
B = {2, 4, 6, 8, 10, 12}

c) Elementary events as union of elementary outcomes

$\{A = 2\} = \{(1,1)\}$
$\{A = 3\} = \{(1,2) \cup (2,1)\}$
$\{A = 4\} = \{(1,3) \cup (2,2) \cup (3,1)\}$
$\{A = 5\} = \{(1,4) \cup (2,3) \cup (3,2) \cup (4,1)\}$
$\{A = 6\} = \{(1,5) \cup (2,4) \cup (3,3) \cup (4,2) \cup (5,1)\}$
$\{A = 7\} = \{(1,6) \cup (2,5) \cup (3,4) \cup (4,3) \cup (5,2) \cup (6,1)\}$
$\{A = 8\} = \{(2,6) \cup (3,5) \cup (4,4) \cup (5,3) \cup (6,2)\}$

$\{A = 9\} = \{(3,6) \cup (4,5) \cup (5,4) \cup (3,3)\}$
$\{A = 10\} = \{(4,6) \cup (5,5) \cup (6,4)\}$
$\{A = 11\} = \{(5,6) \cup (6,5)\}$
$\{A = 12\} = \{(6,6)\}$

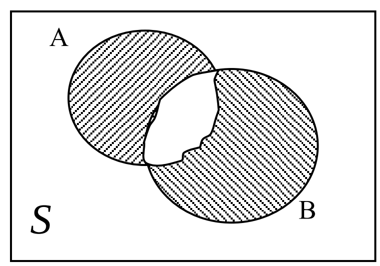
#2.8



$A = (-\infty, r);\ B = (-\infty, s);\ \ r \le s \quad C = (r, s) = ?$

From graph $A \cup C = B$ and since A & C are disjoint $A \cap C = \varnothing$

#2.12 C = Exactly one of A and B occurs
$C = (A \cap B^C) \cup (B \cap A^C)$



#2.18 S = {a, b, c}
P(a, c) = 5/8; P(b, c) = 7/8
Find P(a), P(b), P(c).
1.) P(a, c) = P(a) + P(c) = 5/8
2.) P(b, c) = P(b) + P(c) = 7/ 8
But: 3.) P{a, b, c} = P(S) = 1 = P(a) + P(b) + P(c)
From (1) and (3) we find: P(b) = 3/8
Then from (2) we get: P(c) = 4/8
Finally, P(a) =1/8

#2.21 Show that
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = P[(A \cup B) \cup C)] = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$ Corre. 5
$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap B) \cup (B \cap C)]$ Cor 5 and Distr.
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$
$+ P[(A \cap B) \cap (B \cap C)]$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

#2.25 Use Corr. 7 to prove following:

a.) $P(A \cup B \cup C) \le P(A) + P(B) + P(C)$

Corr. 7: $P(A \cup B) \le P(A) + P(B)$

then

$$P(A \cup B \cup C) \le P(A \cup B) + P(C) \le P(A) + P(B) + P(C)$$

b.) Union Bound:

$$P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$$

We know from Corr. 6

$P(A_1 \cup A_2) \le P(A_1) + P(A_2)$ Case for n = 2

Mathematical Induction: Suppose it is true for n now show it is also true for n+1

Case for n:

$$P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$$

Case for n + 1:

$$P\left(\bigcup_{k=1}^{n+1} A_k\right) = P\left(\bigcup_{k=1}^{n} A_k \cup A_{n+1}\right) \le \sum_{k=1}^{n} P(A_k) + P(A_{n+1}) = \sum_{k=1}^{n+1} P(A_k)$$

$$\therefore \quad P\left(\bigcup_{k=1}^{n+1} A_k\right) \le \sum_{k=1}^{n+1} P(A_k) \quad Q.E.D.$$

#2.26 n-characters typed with Prob(incorrect character) = p

Find: Upper bound on errors in doc?

Let A_i = i^{th} char in error

$$P\{(any\ error\ in\ doc.)\} = P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i) = np$$

#2.32 A combination lock with 3 numbers {0, 1, …, 59}

N = possible combinations?

$= (60)(60)(60) = 60^3 = 6^3(1000) = 216{,}000$

#2.33 Die, coin, card from a normal deck

N = possible outcomes?

$= (6)(2)(52) = 624$

#2.37 10 students occupy 10 desks? 12 desks?
  1^{st} has 10 choices
  2^{nd} has 9 choices

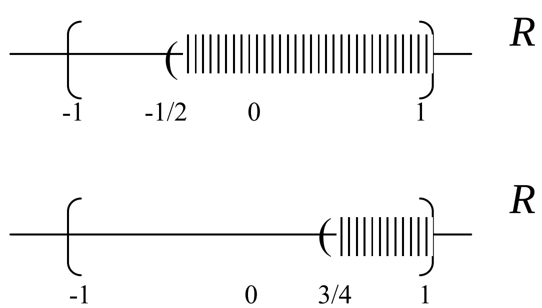
$\Rightarrow$ 10 st. and 10 desks = 10! = 10·9·8·7·6·5·4·3·2·1 = 3,628,800

$\Rightarrow$ 10 st. and 12 desks = 1^{st} 12 choices, 2^{nd} 11 choices, …
  = 12·11·10·9·8·7·6·5·4·3·2·1÷ 2 = 12! ÷ 2 = (3,628,800)(11)(12) ÷ 2
  = 239,500,800 choices

#2.45 P(sum of 3 tosses of a die = 7) = ?

3 tosses of a die : 6^3 = 216 choices
Sum = 7 = {(1,1,5), (1,2,4), (1,3,3), (2,2,3)}
  {1,1,5}: {(1,1,5), (1,5,1), (5,1,1)} = 3 outcomes
  {1,2,4}: {(1,2,4), (1,4,2), (2,1,4), (2,4,1), (4,1,2), (4,2,1)} = 6 outcomes
  {1,3,3}: {(1,3,3), (3,1,3), (3,3,1)} = 3 outcomes
  {2,2,3}: {(2,2,3), (2,3,2), (3,2,2)} = 3 outcomes
$\therefore$ 3 + 6 + 3 + 3 = 15 outcomes yielding "7"
P(sum = 7) = $15/6^3$ = 15/216

#2.52



x : [-1,1] and B = {|x-1/2| < 1}

-1 < x-1/2 < 1 $\Rightarrow$ -1/2 < x < **1**

Find P(B | C) and P(C | B).

$$P(B|C)=\frac{P(B\cap C)}{P(C)}=\frac{P(C)}{P(C)}=1 \quad \text{and} \quad P(C|B)=\frac{P(B\cap C)}{P(B)}=\frac{P(C)}{P(B)}=\frac{1/8}{3/4}=1/6$$

#2.61 Chips from 3 sources A, B, C with Pdef: 0.001, 0.005, 0.01, respectively.
  What is Pdef(A) = ? Pdef(C) = ?

  P(def chip) = P(def/A)P(A) + P(def/B)P(B) + P(def/C)P(C)
  = 10^{-3}P(A) + 5x10^{-3}P(B) = 10x10^{-3}P(C)

$$P(A/def.chip)=\frac{P(def/A)P(A)}{P(def.chip)}=\frac{10^{-3}P(A)}{10^{-3}[P(A)+5P(B)+10P(C)]}=\frac{P(A)}{[P(A)+5P(B)+10P(C)]}$$

Similarly,

$$P(C/def.chip) = \frac{10P(C)}{[P(A)+5P(B)+10P(C)]}$$

If P(A) = P(B) = P(C) = 1/3 , then

$$P(A/def.chip) = \frac{1/3}{(1/3)[1+5+10]} = \frac{1}{16}$$

$$P(C/def.chip) = \frac{(10)(1/3)}{(1/3)[1+5+10]} = \frac{10}{16}$$

$$P(B/def.chip) = \frac{5}{16}$$

#2.69 "Systems are UP" with two controllers. P(UP) = ?

P(System UP) = P(at least one C UP)P(at least 2 P)
P(at least 1 C) = 1-P(both down) = $1-p^2$

P(System UP) = $(1-p^2)[(1-a)^3 + 3(1-a)^2a]$

Numerical case for p = 0.2 and a = 0.1
P(System UP) = $(1-0.04)[(0.9)^3 + (3)(0.1)(0.9)^2] = (0.96)(0.972) = 0.93312$

Which is an improvement over 77.8% of Ex 2.32.

#2.71 100 bits over a BSC with $P_b = 10^{-3}$ What is P(3 or more errors) = ?

$$\text{P(errors} \geq 3) = 1 - P(2\ or\ \ fewer\ errors)$$

$$= 1 - \sum_{k=0}^{2} \binom{100}{k} P_b{}^k (1-P_b)^{100-k} \qquad P_b = 10^{-3}$$

$$= 1 - [(1-P_b)^{100} + 100(1-P_b)^{99} P_b + 100 \cdot 99(1-P_b)^{98} P_b^2]$$

$$= 1 - 0.99985 = 1.5x10^{-4}$$

Marginal for most systems.