Chapter 6: Two-D and Multi-D Sampling Theory and Reconstruction

Ideal 2-D Rectangular Sampling and Reconstruction

Spatial-Domain Model: $Comb(x_1, x_2) = \sum_{n_1}\sum_{n_2} \delta(x_1 - n_1.\Delta x_1, x_2 - n_2.\Delta x_2)$

$$\begin{aligned} f_S(x_1,x_2) &= f_{Continuous}(x_1,x_2).Comb(x_1,x_2) \\ &= \sum_{n_1}\sum_{n_2} f_{Continuous}(n_1.\Delta x_1, n_2.\Delta x_2).\delta(x_1 - n_1.\Delta x_1, x_2 - n_2.\Delta x_2) \\ &= \sum_{n_1}\sum_{n_2} f_{Continuous}(n_1, n_2).\delta(x_1 - n_1.\Delta x_1, x_2 - n_2.\Delta x_2) \end{aligned}$$

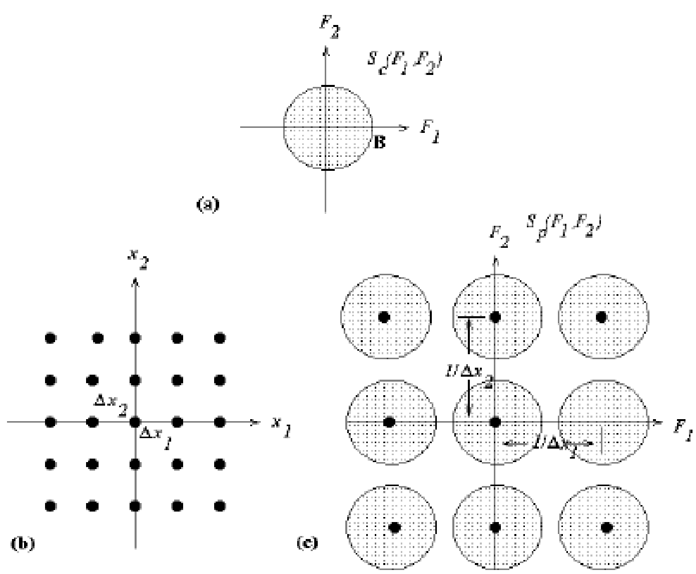
Frequency-Domain Model: $F\{Comb(x_1,x_2)\} = \frac{1}{\Delta x_1.\Delta x_2}\sum_{k_1}\sum_{k_2} \delta(u_1 - \frac{2\pi k_1}{\Delta x_1}, u_2 - \frac{2\pi k_2}{\Delta x_2})$

$$\begin{aligned} F_S(u_1,u_2) &= F_{Continuous}(u_1,u_2) ** F\{Comb(x_1,x_2)\} \\ &= \frac{1}{\Delta x_1.\Delta x_2}\sum_{k_1}\sum_{k_2} F_{Continuous}(u_1,u_2) ** \delta(u_1 - \frac{2\pi k_1}{\Delta x_1}, u_2 - \frac{2\pi k_2}{\Delta x_2}) \\ &= \frac{1}{\Delta x_1.\Delta x_2}\sum_{k_1}\sum_{k_2} F_{Continuous}(u_1 - \frac{2\pi k_1}{\Delta x_1}, u_2 - \frac{2\pi k_2}{\Delta x_2}) \end{aligned}$$

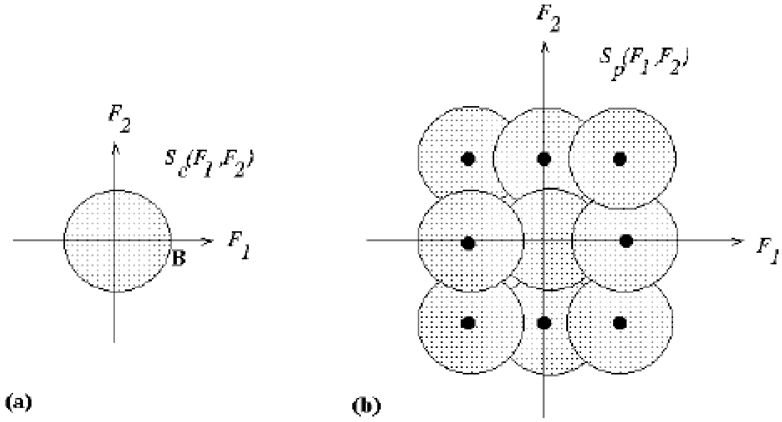
The spectrum of the sampled image is periodic with the horizontal and vertical periods $2\pi/\Delta x_1$ and $2\pi/\Delta x_2$ in radians/length. Assuming that $F_S(u_1,u_2)$ is band-limited with (Ω_1,Ω_2), that is:

$$F_S(u_1,u_2) = \{0 \quad for \quad \Omega_1 > 2\pi B_1 \quad and \quad \Omega_2 > 2\pi B_2\}$$

where $\{B_1,B_2\}$ are *bandwidths* in each direction. Then the **Nyquist Rate** for two-D rectangular sampling is given by: $\Delta x_1 < 1/2B_1$ and $\Delta x_2 < 1/2B_2$.



If the Nyquist rate is not satisfied, then sampled signal spectrum suffers from usual aliasing or spectrum foldover as in the one-D case.



In real-life image acquisition systems, we need to employ anti-aliasing filtering to prevent spectral foldover during the rest of the processing. This is normally achieved by a continuous-domain low-pass filter to bandlimit the image under consideration to the rectangle: $[\frac{-1}{\Delta x_1},\frac{+1}{\Delta x_1}]\,\mathrm{x}\,[\frac{-1}{\Delta x_2},\frac{+1}{\Delta x_2}]$.

Sampling with Finite Aperture:

Space Domain Model: A real–life sensor has a finite size, which results in averaging of image intensity values over a finite area, often referred as a finite sampling aperture. Sampling with a finite aperture is modeled by:

$$f(n_1\Delta x_1, n_2\Delta x_2) = \int\int f(v_1,v_2).h_a(x_1 - v_1, x_2 - v_1)\big|_{\substack{x_1=n_1\Delta x_1\\ x_2=n_2\Delta x_2}} .dv_1 dv_2$$

where $h_a(x_1,x_2)$ denotes the finite aperture function, which can also be expressed in terms of the two-D Sifting Theorem:

$$\begin{aligned} f_a(x_1,x_2) &= [f_{Continuous}(x_1,x_2) **h_a(x_1,x_2)].Comb(x_1,x_2) \\ &= \sum_{n_1}\sum_{n_2} f_{Continuous}(n_1,n_2).\delta(x_1 - n_1.\Delta x_1, x_2 - n_2.\Delta x_2 \end{aligned}$$

Similarly, in the frequency domain:

$$\begin{aligned} F_a(u_1,u_2) &= F_{Continuous}(u_1,u_2) **F\{Comb(x_1,x_2)\} \\ &= \frac{1}{\Delta x_1.\Delta x_2}\sum_{k_1}\sum_{k_2} F_{Continuous}(u_1 - \frac{2\pi k_1}{\Delta x_1}, u_2 - \frac{2\pi k_2}{\Delta x_2}).H_a(u_1 - \frac{2\pi k_1}{\Delta x_1}, u_2 - \frac{2\pi k_2}{\Delta x_2}) \end{aligned}$$

Hence, the effect in the frequency domain is to replicate smoothed versions of the original spectrum.

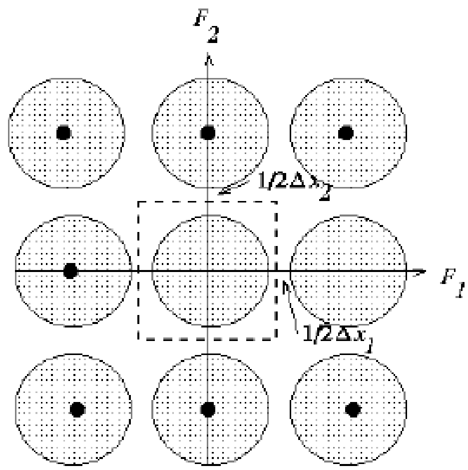
Reconstruction from Samples: An analog function can be reconstructed from the samples using

$$f_{Recons}(x_1, x_2) = \sum_{n_1}\sum_{n_2} f(n_1,n_2).h_i(x_1 - n_1\Delta x_1, x_2 - n_2\Delta x_2)$$

where $h_i(x_1, x_2)$ is the analog **interpolation kernel**.

The reconstructed analog signal is equal to the original analog signal under assumptions:

1. There is no aliasing; i.e., the spectrum of the original continuous domain image is confined within the rectangle $[\frac{-1}{\Delta x_1},\frac{+1}{\Delta x_1}]\,\mathrm{x}\,[\frac{-1}{\Delta x_2},\frac{+1}{\Delta x_2}]$.



2. Reconstruction is performed using an ideal reconstruction filter, which is unrealizable, where the ideal reconstructed image, in the frequency domain, is given by:

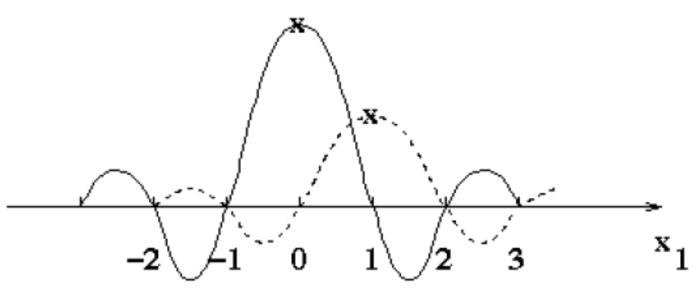
$$F_{\mathrm{Re}cons}(u_1,u_2)=\begin{cases}\Delta x_1.\Delta x_2.F(u_1\Delta x_1, u_2\Delta x_2) & for \quad |u_1|<\dfrac{2\pi}{\Delta x_1}\ and\ |u_2|<\dfrac{2\pi}{\Delta x_2}\\ 0 & Elsewhere\end{cases}$$

If we take the inverse Fourier transform of the above ideal reconstructed image we obtain:

$$f_{\mathrm{Re}cons}(x_1,x_2)=\sum_{n_1}\sum_{n_2} f(n_1,n_2).\frac{Sin[\frac{\pi}{\Delta x_1}(x_1-n_1\Delta x_1)]}{\frac{\pi}{\Delta x_1}(x_1-n_1\Delta x_1)}\frac{Sin[\frac{\pi}{\Delta x_2}(x_2-n_2\Delta x_2)]}{\frac{\pi}{\Delta x_2}(x_2-n_2\Delta x_2)}$$

Therefore, the impulse response of the ideal reconstruction filter is simply:

$$h_i(x_1,x_2)=\frac{Sin[\frac{\pi}{\Delta x_1}(x_1-n_1\Delta x_1)]}{\frac{\pi}{\Delta x_1}(x_1-n_1\Delta x_1)}\frac{Sin[\frac{\pi}{\Delta x_2}(x_2-n_2\Delta x_2)]}{\frac{\pi}{\Delta x_2}(x_2-n_2\Delta x_2)}$$



It is the well-known ideal reconstruction process in the one-D Nyquist sampling. Here the similar process takes place also for the second dimension. The ideal bandlimited interpolation benefits from the periodic zero crossings of the sinc function. However, the sinc function is infinite-extent, which implies that interpolation for each x value requires all of the available samples. Although both assumptions are unrealizable, it is possible to obtain a very close approximation to the original image in many cases.

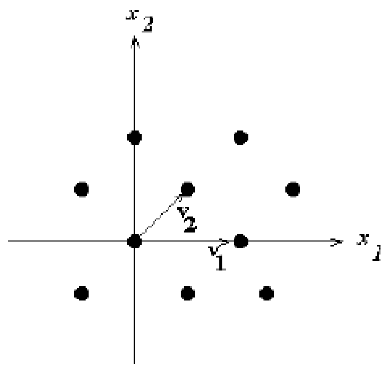
Generalized 2-D Sampling and Reconstruction

Sampling on a Lattice: A lattice Λ^M in $\Re^M$ is the set of all linear combinations of M linearly independent vector in $\Re^M$.

$$\Lambda^M = \{n_1 v_1 + n_2 v_2 + \cdots + n_M v_M \mid n_1, n_2, \cdots, n_M \in Z\} \quad Z\text{: Integers}$$

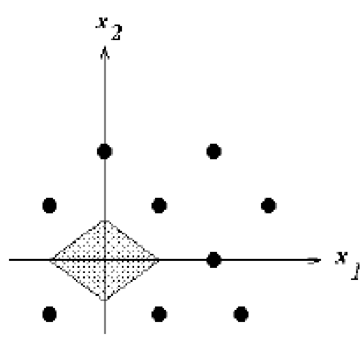
$V = [v_1, v_2, \cdots, v_M]$ is called the sampling matrix and $|\det(V)|$ is the reciprocal of the sampling density. Sampling matrix V is not unique for a given sampling grid, but $\det(V)$ is unique. Sampled signal under these conditions is given by:

$$s(n) = s_C(v_n),\ n \in Z^M = s_C(x),\ x \in \Lambda^M \}$$



Spectrum of a Sampling on a Lattice:

Reciprocal Lattice: The set of all vectors r such that $r^T x$ is an integer for all $x \in \Lambda$ is called the reciprocal lattice Λ^*. A basis for Λ^* is the set of vectors $u_1, u_2, \cdots, u_M$ such that $u_i^T v_j = \delta_{ij}$ for all values of $i, j: 1, 2, \cdots, M$ or in a matrix notation: $U^T V = I$. This is normally explained in terms of unit cell or Voronoi cell, which are the set of points that are closer to the origin than to any other sample point.



In this scenario, the spectrum of the sampled signal is equal to an infinite sum of copies of the analog spectrum shifted according to the reciprocal lattice Λ^*.

$$S_{Sampled}(F) = \frac{1}{|\det(V)|} . \sum_k S_c(F - U.k)$$

Equivalently:

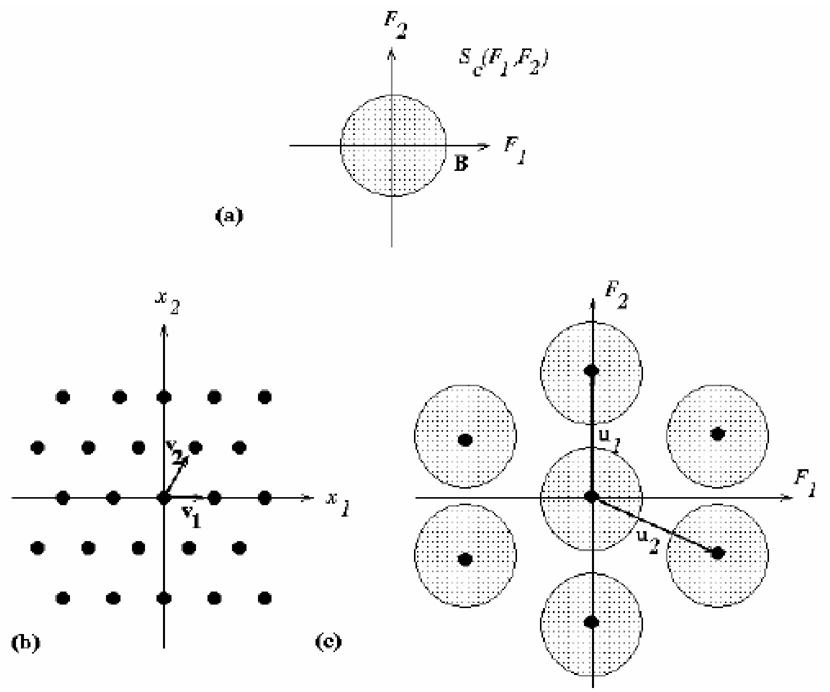
$$S_{Sampled}(F) = \frac{1}{|\det(V)|} . \sum_k S_c(U(F - k)$$

where the periodicity matrix $U = [u_1, u_2, \cdots, u_M]$ satisfies $U^T V = I$ and $u_1, u_2, \cdots, u_M$ are the periodicity vectors.

- These expressions are also valid for rectangular sampling with **V** and **U** being diagonal matrices, and reduce to the case of rectangular sampling.

Sampling operation does not result in loss of information if the analog signal spectrum $S_c(\mathbf{F})=0$ outside the unit cell of Λ^{M*}. With these the generalized sampling problem can be stated as:

1. Given the spectral support $S_c(\mathbf{F})$ of a bandlimited analog signal, select a lattice Λ^{M} or equivalently a sampling matrix **V**; or
2. Given a sampling lattice Λ^{M} design an anti-aliasing filter that confines the spectrum $S_c(\mathbf{F})$ within a unit cell (Voronoi region) of Λ^{M*}.

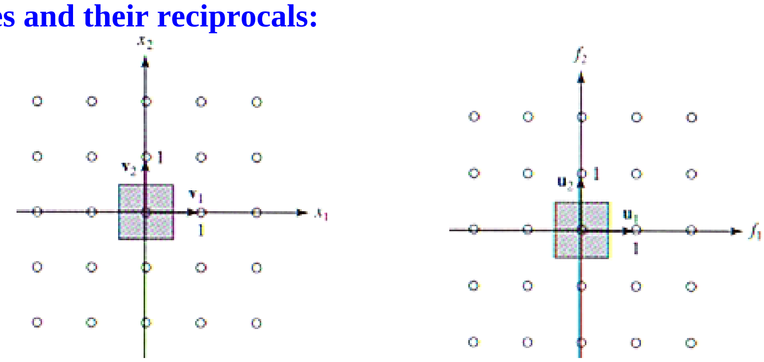


Reconstruction from Samples on a Lattice: A continuous signal can be reconstructed from the samples on Λ^{M} via ideal low-pass filtering over a unit cell of Λ^{M*}.

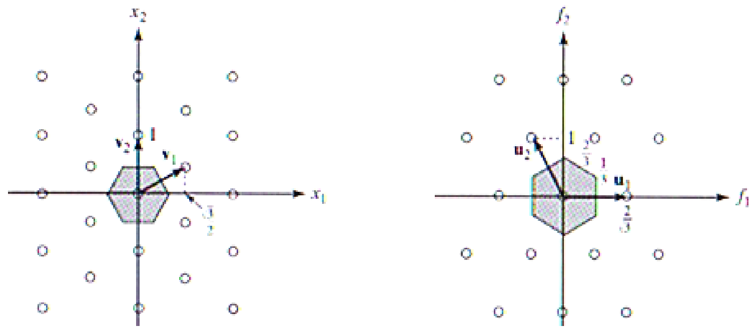
$$S_{\mathrm{Re}\,cons}(F) = \begin{cases} |\det(V)|.S(V^T F) & for \quad F \in P \\ 0 & Elsewhere \end{cases}$$

$$s_{\mathrm{Re}\,cons}(x) = \sum_n s(n).h_i(x - Vn) \text{ with } h_i(x) = |\det(V)| \int_P \exp\{j2\pi F^T x\} dF$$

Examples of lattices and their reciprocals:

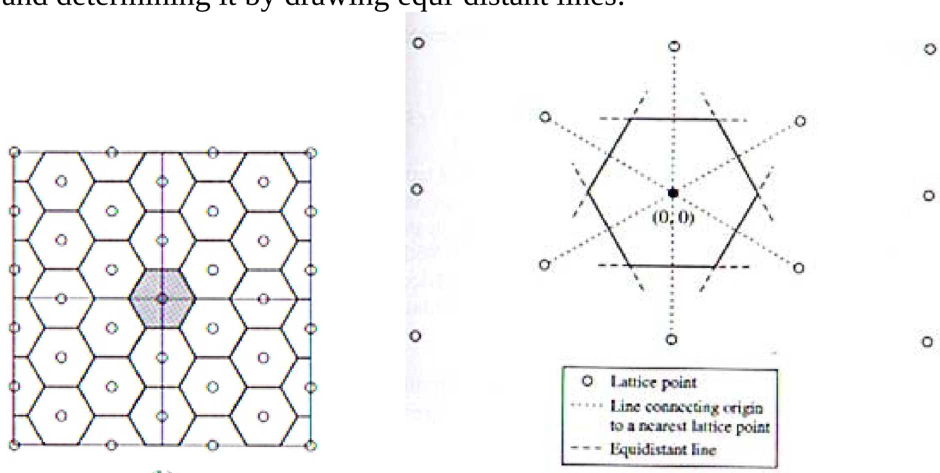


Rectangular Lattice (spatial-domain) and its reciprocal (frequency-domain).



Hexagonal Lattice (spatial-domain) and its reciprocal (frequency-domain).

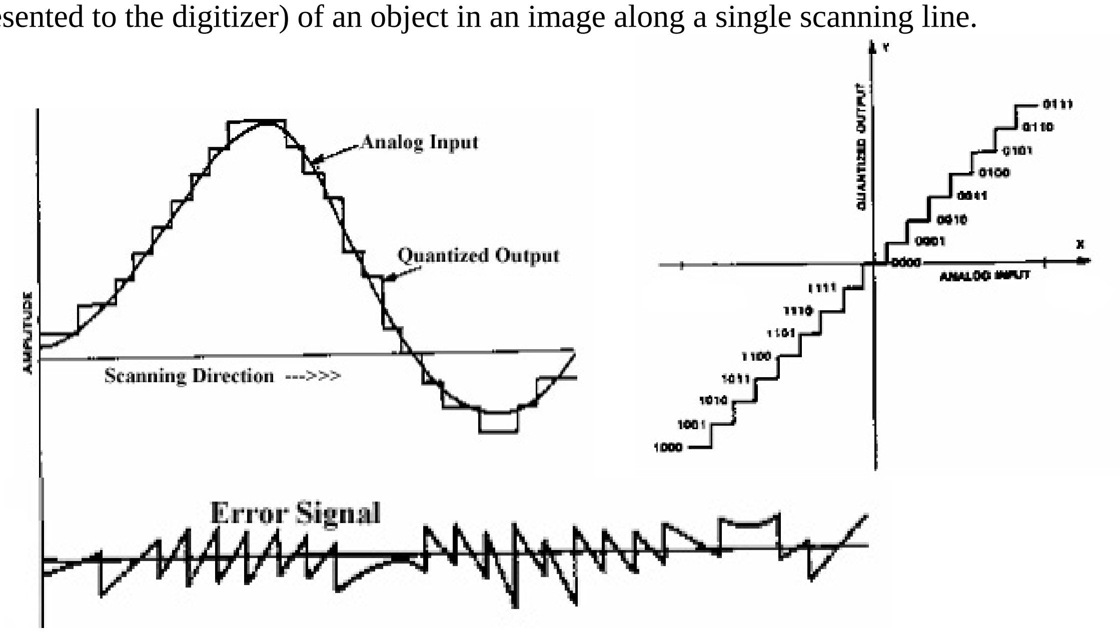
Voronoi Cell and determining it by drawing equi-distant lines:



Quantization

Image amplitude values at each sample are quantized for a finite precision representation. Typically, we use 8 bits per pixel/per color (i.e., 24 bits per pixel for a color image) to represent images. Some applications, such as post-production editing of motion pictures, employ 10 bits/pixel/per color.

Uniform Quantization and Coding: Consider a profile (per color intensity voltage level (R,G,B) presented to the digitizer) of an object in an image along a single scanning line.



Performance is measured in bit rate in bits/sample and the Signal-to-Distortion Ratio (**SDR**), commonly known as **SNR**.

SNR for Uniform Quantizers: If

- Coding rate in bits per sample: R
- Maximum amplitude level of pixels: $X_{\max}$
- Variance or power of input pixels: σ_X^2

$$SNR = 10.\log_{10}\left(\frac{\sigma_x^2 .3.2^{2R}}{x_{\max}^2}\right)$$

and if $x_{\max} = 4.\sigma_x$, i.e., "four sigma loading" in the telecom jargon:

$$SNR = 6.02 * R - 10.\log_{10}(f_l^2/3) \quad dB$$

where $f_l = \frac{x_{\max}}{\sigma_x} = 4$ $\quad\Longrightarrow\quad$ $SNR = 6.02R - 7.27\ dB$

Non-Uniform Quantizers:

- Step sizes are variable.
- They are more closely spaced for samples which are frequent.
- Very infrequent samples are bundled together or even some neglected totally.
- Step sizes are chosen either:
  - Logarithmically as in Log-PCM or
  - According to some optimization principle. Two classes:
    1. Lloyd II - Max quantizers. Optimized from empirical statistical assumptions, i.e., Gaussian, exponential, Gamma distribution for samples.
    2. Lloyd I quantizers: Designed by finding centroids of the data at hand.

Adaptive Quantizers:

Uniform and non-uniform quantizers with fixed quantization levels can overflow and underflow easily if the input signal levels change over time. In other words, a large number of higher index levels are not used at all for low amplitude signals. In the case of signals with large amplitude ranges, the lower indices are never transmitted. The remedy to this situation is to adapt the quantizer levels to the dynamics of the input signal.

Adaptation could be in two ways:

1. **Backward adaptation:** Quantize the current signal; estimate the levels for the next iteration; quantize the new signal with the estimate from the previous iteration.
2. **Forward adaptation:** Buffer the signal; estimate the levels and quantize with a small delay.

Some of these ideas will be discussed within the framework of individual coding schemes.