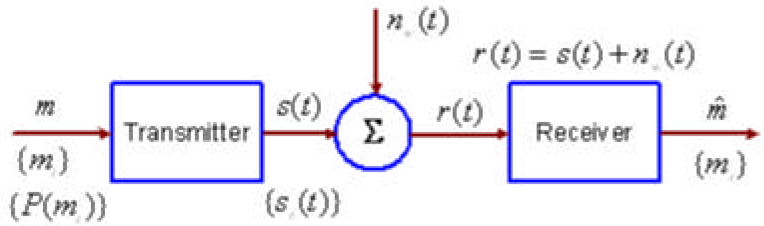
2. OPTIMUM RECEIVER PRINCIPLES

2.1 Maximum *Aposteriori* Receiver

Consider the generic block diagram of end-to-end communication over the ubiquitous additive white Gaussian noise (AWGN) channel.



- Source: $\{m_i\}$ with *apriori* probabilities: $\{P(m_i)\}$
- Transmitter: A particular message symbol is represented by a signal waveform allowable in the signal space permitted for a given modulation technique.

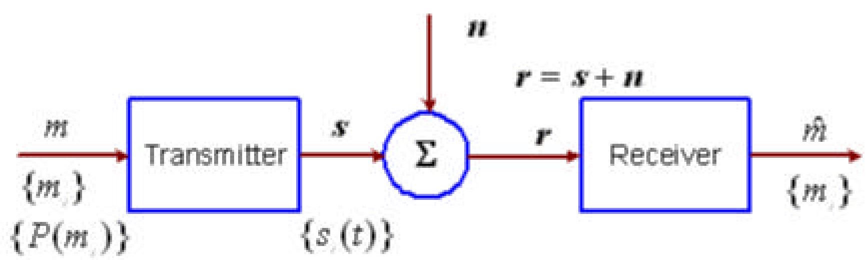
$$m = m_i \Leftrightarrow s(t) = s_i(t) \quad (2.1)$$

- Channel: $r(t) = s(t) + n_w(t)$ (2.2)

Problem 1: Design an optimum receiver which estimates $\hat{m}$ for the transmitted signal $s(t)$ of a source output m such that the probability of error $P(e) \equiv \Pr ob(\hat{m} \neq m)$ is MINIMUM.

Problem 2: Given that $\{P(m_i)\}$ are UNKNOWN, which is the real-life problem in many emerging communication systems, design a similar optimum receiver. (Inherently more difficult task!).

VECTOR CHANNEL: Consider the case when a sequence of source outputs are bundled into a vector form and transmitted as the case of QAM and other m-ary signaling schemes. In some cases, the signal itself may be in a vector form to start with, as in the case of LPC coefficients in a CELP Speech coder.



- Source Information is mapped into source vectors: $\{\underline{s}_i; i = 0,1,\ldots,M-1\}$, where each vector is composed of N-components: $\underline{s}_i = [s_{i1}, s_{i2},\ldots, s_{iN}]$.
- Received Information is also mapped into vectors:

$$\underline{r}_i = \underline{s}_i + \underline{n}_i = [s_{i1} + n_{i1}, s_{i2} + n_{i2},\ldots, s_{iN} + n_{iN}] \quad (2.3)$$

Given that the received vector is a point $\underline{r} = \underline{r}$ in the N-dimensional space with coordinates:

$\underline{?} = [?_1, ?_2, \ldots, ?_N]$ then the optimum receiver must compute the transmitted vector signal $\underline{s}_i$ for the message m_i having a maximum aposteriori probability from its knowledge of the set of parameters: $P_{\underline{r}|\underline{s}}$, $\{\underline{s}_i\}$, and the source distribution: $\{\Pr ob(m\ )\}$.

In other words:

$$\hat{m} = m_k \quad \text{if} \quad \Pr ob(m_k \mid \underline{r} = \underline{r}) > \Pr ob(m_i \mid \underline{r} = \underline{r}) \quad \textit{for all } i \neq k \tag{2.4}$$

which is a nearly impossible challenge to meet in many real-life situations.

Do we have an equivalent task?

Consider the correct decision for a given incoming vector:

$$\Pr ob(C \mid \underline{r} = \underline{r}) = \Pr ob(m_k \mid \underline{r} = \underline{r}) \tag{2.5}$$

and the overall correct decision is simply ensemble of correct decisions:

$$\Pr ob(C) = \int_{-\infty}^{\infty} \Pr ob(C \mid \underline{r} = \underline{r}).P_{\underline{r}}(\underline{r})d\,\underline{r} \tag{2.6}$$

Since $P_{\underline{r}}(\underline{r}) \geq 0$ we do not need to include it in the maximization process, i.e. only the term $\Pr ob(C \mid \underline{r} = \underline{r})$ must be maximized. Let us use the Bayes Rule on (2.5)

$$\Pr ob(m_i \mid \underline{r} = \underline{r}) = P(m_i).\Pr ob_{\underline{r}}(\underline{r} \mid m_i) / P_{\underline{r}}(\underline{r}) \tag{2.7}$$

but the statement $m = m_i$ is equivalent to $\underline{s} = \underline{s}_i$ which implies:

$$\Pr ob_{\underline{r}}(\underline{r} \mid m_i) = \Pr ob_{\underline{r}}(\underline{r} \mid \underline{s} = \underline{s}_i) \tag{2.8}$$

Furthermore, the denominator term is independent of the index i, hence, the maximization and we have the revised principle for our optimum receiver:

$$\hat{m} = m_k \quad \text{if} \quad P(m_i).\Pr ob_{\underline{r}}(\underline{r} \mid \underline{s} = \underline{s}_i) \text{ is maximum when } i = k \tag{2.9}$$

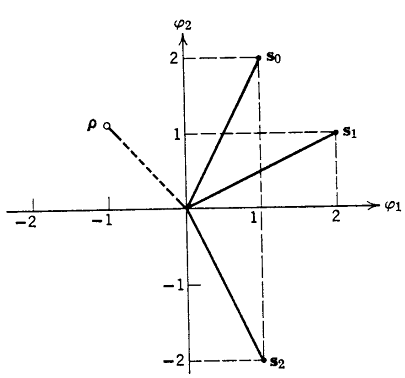
When $P(m_i)$ are not known and the receiver can only maximize the last portion of (2.9). Then we have a restricted version of the general optimum receiver called **MAXIMUM-LIKELIHOOD (ML) Receiver.**

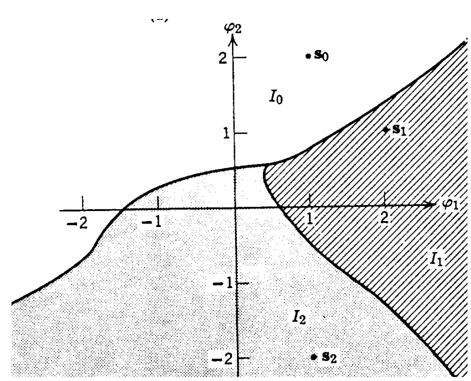
ML Receiver Principle:

$$\hat{m} \Rightarrow m_k \text{ when } \Pr ob_{\underline{r}}(\underline{r} \mid \underline{s} = \underline{s}_i) \quad \text{is MAXIMUM.} \tag{2.10}$$

Decision Regions are needed to perform the mapping properly for each signal vector.

Example 2.1: Given (3) input vectors in a 2-D vector space with the following signal set assignment: $m_0 \Rightarrow \underline{s}_0 = [1,2]$; $m_1 \Rightarrow \underline{s}_1 = [2,1]$; and $m_2 \Rightarrow \underline{s}_2 = [1,-2]$





Let us also assume that the input message probabilities: $P(m_0), P(m_1), P(m_2)$ are given. For this assignment, our receiver will compute:

$$P(m_i).\Pr ob_{\underline{r}}(\underline{r} \mid \underline{s} = \underline{s}_i) \qquad \textit{for } I=0,1,2.$$

An ML Receiver will choose the index of the message with the largest product above.

For every point $\underline{r}$ in (j_1, j_2) plane an assignment can be made if the plane is partitioned into disjoint regions $\{I_i\}$ *for* $i=0,1,2$; which are called decision regions, very similar to the codeword selection process in Vector Quantization (VQ). Then we have the ML receiver as a simple geometric map:

$$\underline{r} = I_k \quad \Rightarrow \quad \hat{m} = m_k \text{ and an error is made if } \hat{m} \Rightarrow m_k \quad iff \quad \underline{r} \not\subset I_k \tag{2.11}$$

2.2 ML Receiver for AWGN Channel

Given that the signal in the channel is corrupted by a zero mean AWGN with a variance s^2.

$$\underline{r} = \underline{s} + \underline{n} = [s_1 + n_1, s_2 + n_2, \ldots, s_N + n_N] \tag{2.12}$$

Now:

$$\underline{r} = \underline{r} \text{ when } \underline{s} = \underline{s}_i \quad iff \quad \underline{n} = \underline{r} - \underline{s}_i \tag{2.13}$$

And then

$$P_{\underline{r}}(\underline{r} \mid \underline{s} = \underline{s}_i) = P_{\underline{n}}(\underline{r} - \underline{s}_i \mid \underline{s} = \underline{s}_i) \qquad for \quad i = 0,1,\ldots,M-1 \tag{2.14}$$

Since the signal $\underline{s}$ and the channel noise $\underline{n}$ are statistically independent $P_{\underline{n}|\underline{s}} = P_{\underline{n}}$. This simplifies (2.14) into:

$$P_{\underline{n}}(\underline{r} - \underline{s}_i \mid \underline{s} = \underline{s}_i) = P_{\underline{n}}(\underline{r} - \underline{s}_i) \tag{2.15}$$

In this case, the general ML decision function becomes $P(m_i).P_{\underline{n}}(\underline{r} - \underline{s}_i)$. Now the components of signal is assumed to be independent, noise has a zero-mean we can write the noise distribution:

$$P_{\underline{n}}(\underline{u}) = \frac{1}{(2p\,s^2)^{N/2}} \exp\{-\frac{1}{2s^2} \sum_{j=1}^{N} u_j^2\} \tag{2.16}$$

Let us use the following dot-product notation:

$$|\underline{u}|^2 = \underline{u} \bullet \underline{u}^* = \sum_{j=1}^{N} u_j^2$$

Our distribution is written as:

$$P_{\underline{n}}(\underline{u}) = \frac{1}{(2p\,s^2)^{N/2}} \exp\{-\frac{1}{2s^2} |\underline{u}|^2\} \tag{2.17}$$

Then for this probability system we have the ML principle as:

$$\hat{m} \Rightarrow m_i \quad whenever \quad P(m_i).\exp\{-\frac{1}{2r^2} |\underline{r} - \underline{s}_i|^2\} \text{ is maximum.} \tag{2.18}$$

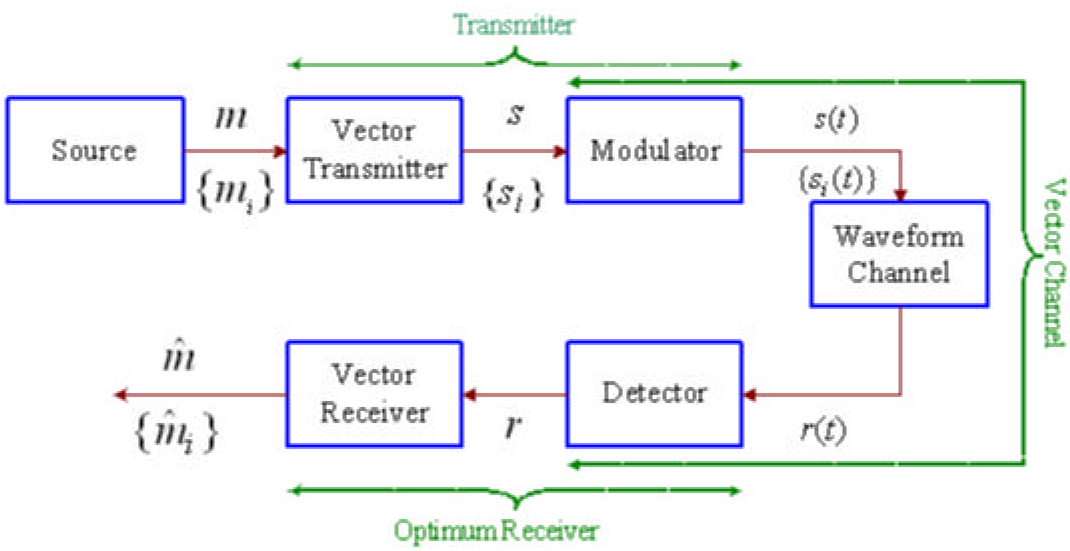
Equivalently, the task is to **MINIMIZE:**

$$|\underline{r} - \underline{s}_i|^2 - (2s^2).\log_e P(m_i) \tag{2.19}$$

The first term is the Euclidean Distance between the received vector and a candidate signal vector. If all the messages are equally likely then the optimum decision rule does not depend on the index at all and we have the **MINIMUM MEAN-SQUARE (MMS) DISTANCE Receiver**. That is we assign the message index of the closest neighbor of the incoming vector, which is also known as the Nearest Neighbor Rule in VQ and other clustering techniques.

2.3 Correlation and Matched-Filter Receivers

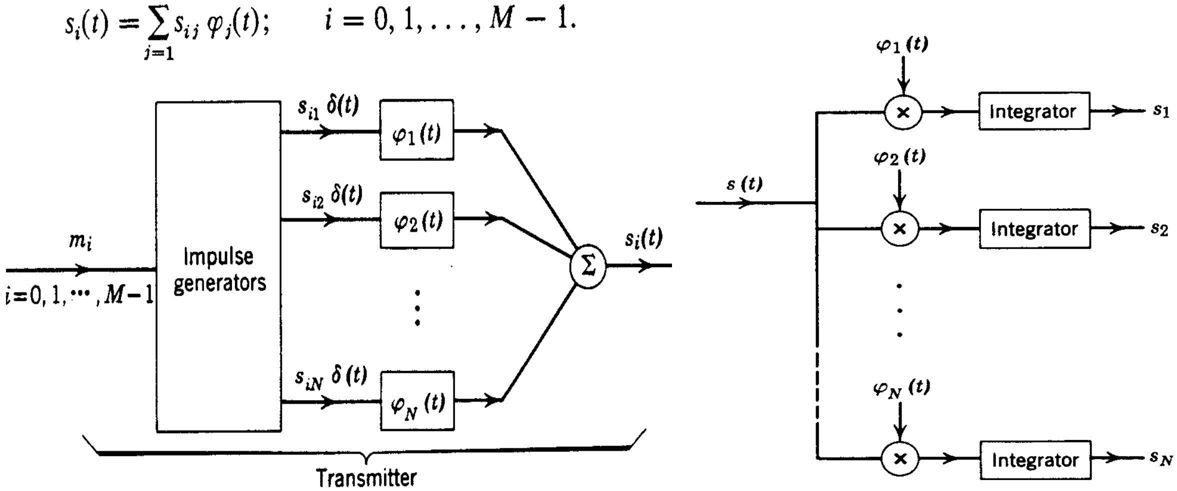
If we revisit the Communication System Block Diagram for vector signals as shown below, it would be necessary to synthesize waveform signals to be transmitted over real-life channels, such as the twisted-pair or coaxial cable, microwave or fiber-optic links.



- It is necessary to synthesize the signal set $\{s_i(t)\}$ at the transmitter. This can be achieved by "building blocks waveforms".
- Synthesis signal sets and Recovery of signal vectors:

1. A set of N integrating filters are used to generate N signal components with strengths $\{s_{ij}\}$.
2. The filter outputs are summed to yield the signal waveform: $s(t)$ to be transmitted for a particular message m_i for each of M different messages.

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\,\varphi_j(t); \qquad i = 0, 1, \ldots, M-1.$$



$$s_i(t) = \sum_{j=1}^{N} s_{ij} j_j(t) \quad for \quad i = 0,1,\ldots,M-1 \qquad (2.20)$$

1. Let us choose the building-block waveforms from an ortho-normal set such that:

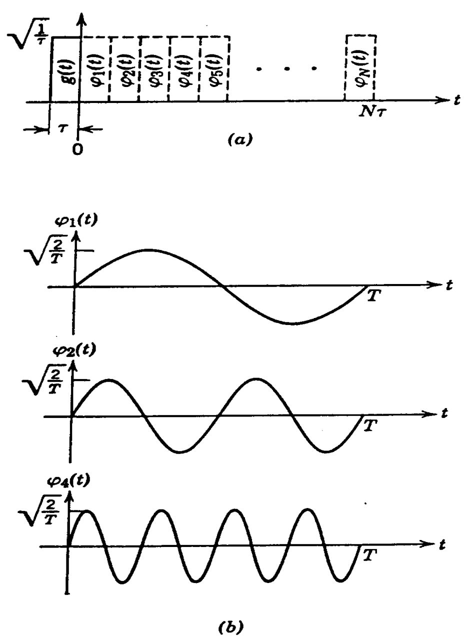
$$\int_{-\infty}^{\aleph} j_j(t) j_l(t) dt = \begin{cases} 1 & if\ j = l \\ 0 & if\ j \neq l \end{cases} \quad for\, all \quad 1 \leq i, j \leq N \qquad (2.21)$$

2. This will yield a probability of error independent of the actual wave-shapes.
3. We can exactly recover the signal vectors and hence, the messages in the absence of channel if we push these synthesized waveforms of (2.20) into a simple integrating filter structure as shown above.

$$\int s_i(t) j_l(t) dt = \int [\sum_{j=1}^{N} s_{ij} j_j(t)] j_l(t) dt = \sum_{j=1}^{N} s_{ij} d_{jl} = s_{il} \qquad (2.22)$$

If we perform similar integration for all the branches we obtain: $\underline{s_i} = [s_{i1}, s_{i2}, \ldots, s_{iN}]$.

4. Examples of Orthonormal Signal Sets:
   - Orthonormal time-shifted pulses: $j_j(t) = g(t - jt) \quad for \ \ j = 1,2,\ldots,N$
   - Orthonormal Fourier Transform pulses: $j_j(t) = \begin{cases} \sqrt{1/t} & -T \leq t < 0 \\ 0 & otherwise \end{cases}$



The optimum ML receiver of the system performs:

$$Set\ \hat{m} = m_k \quad if\ \left|\underline{r} - \underline{s}_i\right|^2 - 2s_n^2 \log P(m_i)\ \ is\ MINIMUM. \qquad (2.23)$$

Square operations can be eliminated in (2.23) by observing:

$$\left|\underline{r} - \underline{s}_i\right|^2 = \left|\underline{r}\right|^2 - 2(\underline{r} \bullet \underline{s}_i) + \left|\underline{s}_i\right|^2 \qquad (2.24)$$

where the dot product is given also by:

$$\underline{r} \bullet \underline{s}_i \equiv \sum_{j=1}^{N} r_j s_{ij} \tag{2.25}$$

Observations :

- Note 1: First term in (2.24) is independent of the index and no need to worry in optimization.

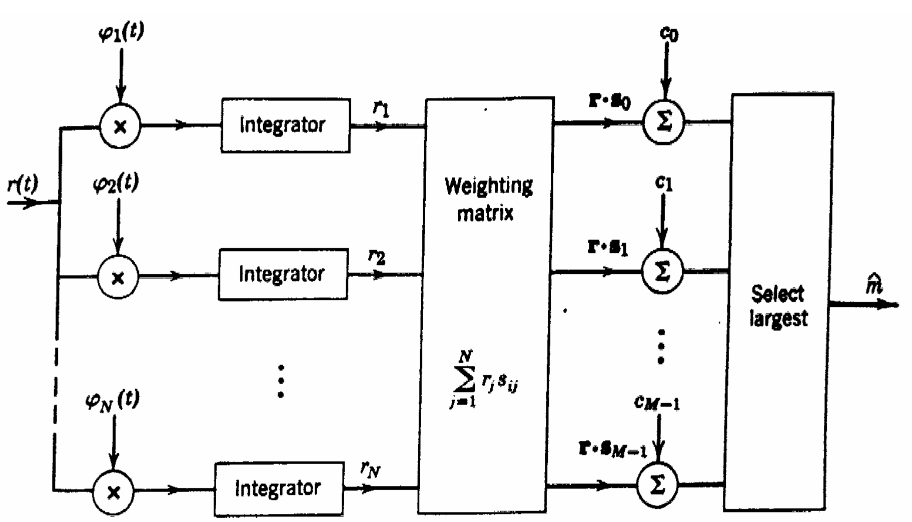
- Note 2: Last terms in (2.23) and (2.24) depend only on source side information supplied by the designer then they can be combined into a constant parameter set and burned into the ROM of the system:

$$c_i \equiv (1/2)[s_n^2 \log P(m_i) - |\underline{s}_i|^2] \tag{2.26}$$

- The optimum receiver of (2.23) is now equivalent to:

$$Set\ \hat{m} = m_k \quad if\ (\underline{r} \bullet \underline{s}_i + c_i)\ is\ MAX. \tag{2.27}$$

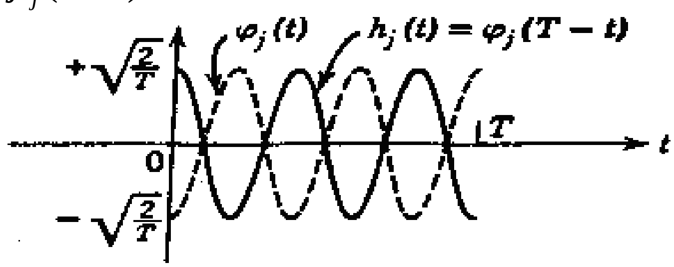
which is simply the structure of a **CORRELATION RECEIVER.**



Note: When the source vocabulary size M is not very large this implementation is not costly and most of the operations to the right of the *"Integrators"* can be done by table look-ups. However, when M is very large then the dot-products are usually handled by using "DSP" based devices. The use of multipliers can be avoided if we replace the structure to the left of the *"Weighting Matrix"* as follows:

1. Let us consider a filter with an impulse response

$$h_j(t) = j_j(T - t). \tag{2.28}$$



2. If the input to this filter is $\underline{r}(t)$ then its response is simply:

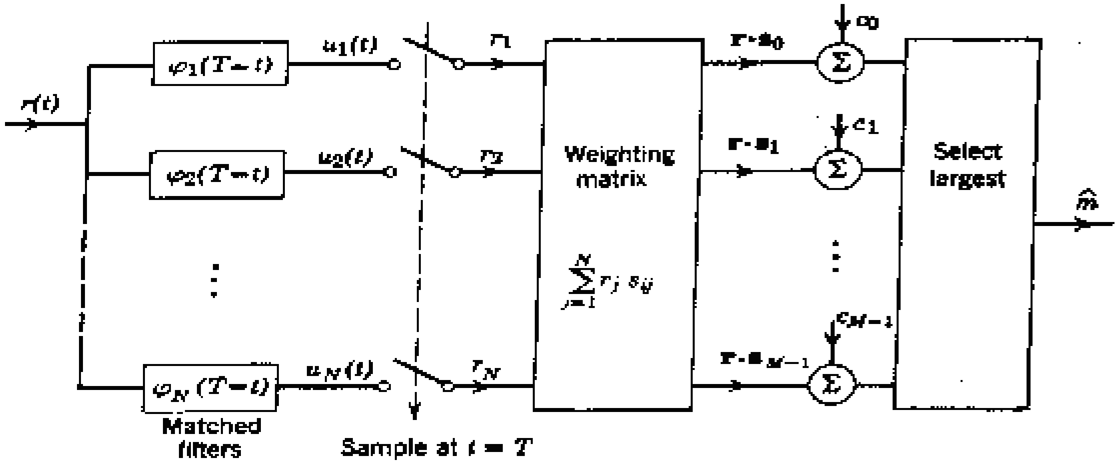
$$u_j(t) = \int_{-\infty}^{\infty} \underline{r}(a) h(t-a) da = \int_{-\infty}^{\infty} \underline{r}(a) j_j(T-t+a) da \qquad (2.29)$$

3. When we sample the output at $t=T$ we have

$$u_j(T) \equiv r_j \qquad (2.30)$$

4. Finally, the task is to push it through the weighting matrix and the rest of the receiver above.

This receiver is called a **"MATCHED-FILTER" Receiver** since it is constructed by using the shifted versions of the signal building block functions: $j_j(T-t)$.



Example 2.2: Consider the case for a gated-sinusoidal tone signal with a gate period of T seconds as shown below. The convolution operation in the matched -filter above will result in a triangular enveloped sinusoid with the same frequency and thus it will peak at the sampling instant T.