Chapter 7: Analysis and Processing of Random Signals

Power Spectral Density (PSD) for Continuous-Time Random Processes:

Let X(t) be a continuous-time WSS random process with mean m_X and an autocorrelation function: $R_X(\tau)$ The Fourier transform gives:

$$S_X(f) = F\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

Remember that the autocorrelation function is an even function of τ:

$$R_X(\tau) = R_X(-\tau)$$

Therefore,

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\{\cos 2\pi f\tau - j\sin 2\pi f\tau\} d\tau = \int_{-\infty}^{\infty} R_X(\tau)\cos 2\pi f\tau d\tau$$

Since integral of even/odd function = 0: $S_X(f)$ is real-valued and an even function of *f*. Also, $S_X(f) \geq 0$ for all *f*

$$R_X(\tau) = F^{-1}\{S_X(f)\} = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

Average Power of *X(t)* across 1-ohm resistor:

$$E\left[X^2(t)\right] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) e^{jo} df = \int_{-\infty}^{\infty} S_X(f) df$$

Since

$$C_X(\tau) = R_X(\tau) - m_X^2$$

$$S_X(f) = \Im\left\{C_X(\tau) + m_X^2\right\} = \Im\{C_X(\tau)\} + m_X^2 \delta(f)$$

Cross-Power Spectral Density

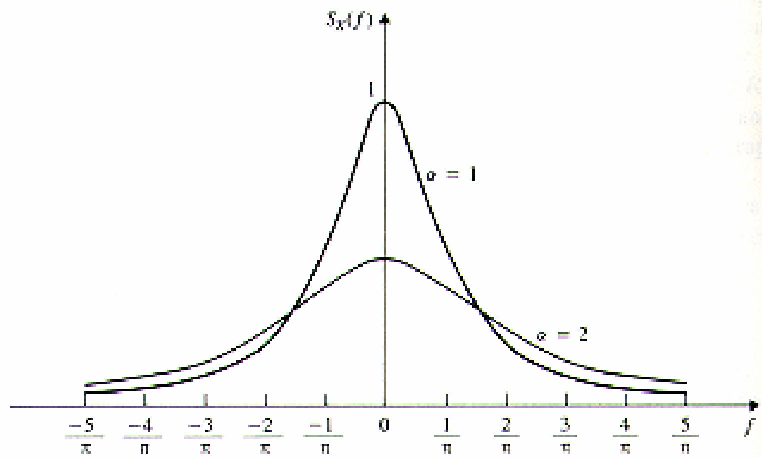
$$S_{XY}(f) = F\{R_{XY}(\tau)\} \qquad \text{where} \quad R_{XY}(\tau) = E[X(t+\tau)Y(\tau)]$$

Example: 7.1 Find Power Spectral Density of Random Telegraph Signal with the autocorrelation function: $R_X(\tau) = e^{-2\alpha|\tau|}$. The Fourier transform gives the PSD:

$$S_X(f) = \int_{-\infty}^{0} e^{2\alpha\tau} e^{-j2\pi f\tau} d\tau + \int_{0}^{\infty} e^{-2\alpha\tau} e^{-j2\pi f\tau} d\tau$$

$$= \frac{1}{2\alpha - j2\pi f} + \frac{1}{2\alpha + j2\pi f} = \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}$$

FIGURE 7.1
Power spectral density of a random telegraph signal with $\alpha = 1$ and $\alpha = 2$ transitions per second.



Above figure shows Power Spectral Density for $\alpha = 1$ and $\alpha = 2$. Note that $\alpha = 2$ has a greater high-frequency content when compared with a smaller α.

Example: 7.2 Given: $X(t) = a\cos(2\pi f_0 t + \theta)$, θ uniformly distributed $(0,2\pi)$, find Power Spectral Density. From Ex: 6.7 we have the autocorrelation function:

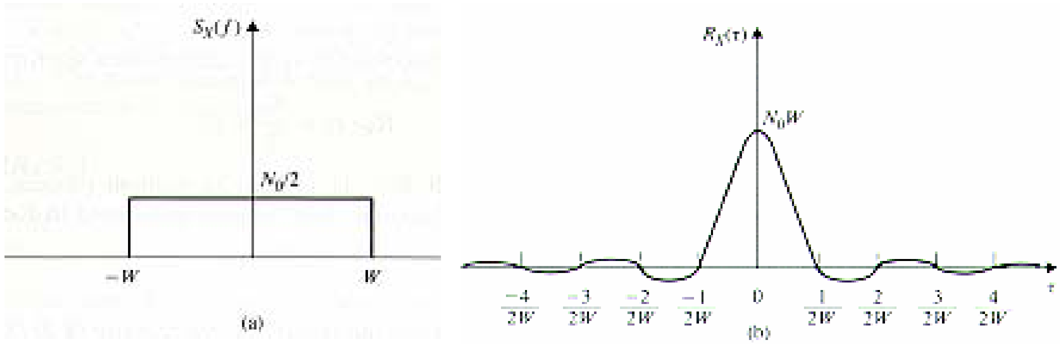
$$R_X(\tau) = \frac{a^2}{2}\cos 2\pi f_0 \tau$$

and the Fourier transform gives:

$$S_X(f) = \frac{a^2}{2} F\{\cos 2\pi f_0 \tau\} = \frac{a^2}{4}\delta(f - f_0) + \frac{a^2}{4}\delta(f + f_0)$$

The signal has average power $R_X(0) = \frac{a^2}{2}$ where all of this power is localized at $\pm f_0$.

Example: 7.3 WSS White Noise in the frequency range: $-W \le f \le W$
White Noise band-limited in the frequency range: $-W \le f \le W$ Hz , i.e. colored (pink) noise. If w is very large then it is approximately white.



Average Power: $E\left[X^2(t)\right] = \int_{-W}^{W} \frac{N_0}{2} df = N_0 W$

Autocorrelation:

$$R_X(\tau) = \frac{1}{2} N_0 \int_{-W}^{W} e^{j2\pi f\tau} df = \frac{1}{2} N_0 \frac{e^{-j2\pi W\tau} - e^{-j2\pi W\tau}}{-j2\pi\tau} = \frac{N_0 \sin(2\pi W\tau)}{2\pi\tau}$$

Note: X(t) and X(t + τ) are uncorrelated at τ = ± k/2W , k = 1, 2, …

Power Spectral Density of White Noise W(t): $S_W(f) = \frac{N_0}{2}$ for all *f*.

As W → ∞, we have: $R_W(\tau) = \frac{N_0}{2}\delta(\tau)$. If W(t) is Gaussian R. P. , then W(t) is White Gaussian Noise, is discussed in Example: 6.38

Example: 7.5 Given: *Y(t) = X(t-d)* where *d* is constant delay and X(t) is WSS, compute the PSD function.

$$R_{YX}(\tau) = E\left[Y(t+\tau)X(t)\right] = E\left[X(t+\tau-d)X(t)\right] = R_X(\tau-d)$$
$$S_{YX}(f) = F\left\{R_X(\tau-d)\right\} = S_X(f)e^{-j2\pi fd}$$
$$= S_X(f)\cos(2\pi fd) - jS_X(f)\sin(2\pi fd)$$

$$R_Y(\tau) = E\left[Y(t+\tau)Y(t)\right] = E\left[X(t+\tau-d)X(t-d)\right] = R_X(\tau)$$
$$\Rightarrow \quad S_Y(f) = F\left\{R_Y(\tau)\right\} = F\left\{R_X(\tau)\right\} = S_X(f)$$

Note: The result: $S_Y(f) = S_X(f)$ does not imply *X(t) = Y(t)*.

Power Spectral Density for Discrete-Time Random Processes:

$$S_X(f) = F\left\{R_X(k)\right\} = \sum_{k=-\infty}^{\infty} R_X(k)e^{-j2\pi fk}$$

where -1/2 ≤ *f* ≤ ½. This is due to $S_X(f)$ being periodic.

$S_X(f) \geq 0$ and even function of *f*. $S_X(f) = S_X(f+1)$

$$R_X(k) = \Im^{-1}\{S_X(f)\} = \int_{-1/2}^{1/2} S_X(f) e^{j2\pi f k}\, df$$

Cross-Power Spectral Density: Assume that X_n, Y_n are jointly WSS

$$S_{XY}(f) = F\{R_{XY}(k)\} \qquad \text{where } R_{XY}(k) = E[X_{n+k} Y_n]$$

Example: 7.6 X_n uncorrelated r.v., zero mean, variance σ^2_X (White Noise Process). Find the PSD function: $S_X(f)$

$$R_X(k) = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and

$$S_X(f) = \sum R_X(k) e^{-j2\pi f k} = \sum \delta(k) \sigma_X^2 e^{-j2\pi f k} = \sigma_X^2 \quad \text{for } -1/2 < f \le 1/2$$

Example 7.7: Given $Y_n = X_n + \alpha X_{n-1}$ Where X_n is white noise process of Ex. 7.6

$$E[Y_n] = E[X_n] + \alpha E[X_{n-1}] = 0$$

$$E[Y_n Y_{n+k}] = E[(X_n + \alpha X_{n-1})(X_{n+k} + \alpha X_{n+k-1})]$$

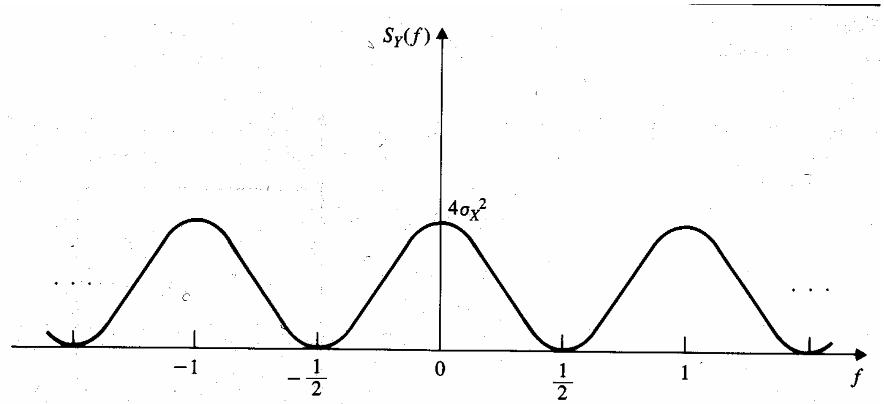
$$= \underbrace{E[X_n X_{n+k}]}_{\begin{array}{ll} \sigma_X^2 & k=0 \\ 0 & k=\pm 1 \\ 0 & o.w. \end{array}} + \underbrace{\alpha E[X_n X_{n+k-1}] + \alpha E[X_{n-1} X_{n+k}]}_{\begin{array}{ll} 0 & k=0 \\ \alpha\sigma_X^2 & k=\pm 1 \\ 0 & o.w. \end{array}} + \underbrace{\alpha^2 E[X_{n-1} X_{n+k-1}]}_{\begin{array}{ll} \alpha^2\sigma_X^2 & k=0 \\ 0 & k=\pm 1 \\ 0 & o.w. \end{array}}$$

$$E[Y_n Y_{n+k}] = \begin{cases} (1+\alpha^2)\sigma_X^2 & k = 0 \\ \alpha\sigma_X^2 & k = \pm 1 \\ 0 & o.w. \end{cases}$$

$$S_Y(f) = F\{E[Y_n Y_{n+k}]\} = (1+\alpha^2)\sigma_X^2 + \alpha\sigma_X^2 \{e^{j2\pi f} + e^{-j2\pi f}\}$$

$$= \sigma_X^2 \{(1+\alpha^2) + 2\alpha \cos 2\pi f\}$$

FIGURE 7.3
Power spectral density of moving average process discussed in Example 7.7.



Power Spectral Density as a Time Average:

Periodogram: Let $X_0, X_1, \ldots, X_{k-1}$ be k observations from a discrete-time WSS Process

$$\hat{x}_k(f) = \sum_{m=0}^{k-1} x_m e^{-j2\pi f m} \qquad \text{DFT of } X_k$$

Periodogram Estimate: $\hat{p}_k(f) = \frac{1}{k}\left|\hat{x}_k(f)\right|^2$

then it can be shown that $E\{\hat{p}_k(f)\} \to S_X(f)$ as $k \to \infty$

If X(t) is a continuous-time WSS process then $\hat{p}_T(f) = \frac{1}{T}\left|\hat{x}_T(f)\right|^2$

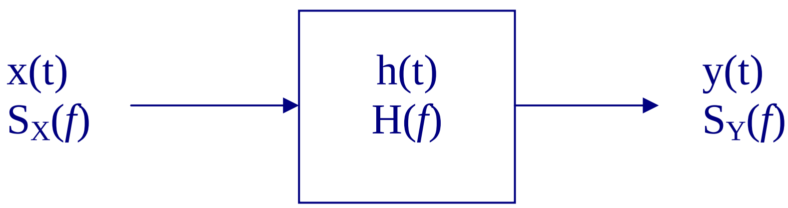
Where

$$\hat{x}_T(f) = \int x(t')\, e^{-j2\pi f t'}\, dt'$$

then it can be shown that

$$E\{\hat{p}_T(f)\} \to S_X(f) \text{ as } T \to \infty$$

Random Signals Through Linear Systems:



Let us recall that a system is linear if zero-in yields zero-out and if the superposition theorem holds:

$$T[\alpha x_1(t)+\beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)]$$

Similarly, a system is time-invariant if $x(t-\tau)$ yields $y(t-\tau)$, then

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(s)x(t-s)ds = \int_{-\infty}^{\infty} h(t-s)x(s)ds$$

where h(t) is the impulse response $h(t) = T[\delta(t)]$.

Transfer function or frequency response of the system:

$$H(f) = F\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi f t}\, dt$$

A system is causal if the response at t depends only on past and present values of the input if h(t) = 0 for t < 0.

For random signal input X(t) , then

$$Y(t) = \int_{-\infty}^{\infty} h(s)X(t-s)ds = \int_{-\infty}^{\infty} h(t-s)X(s)ds$$

If these integrals exist in the mean-square sense and if X(t) is WSS then Y(t) is also WSS.

$$m_Y = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(s)X(t-s)ds\right] = \int_{-\infty}^{\infty} h(s)E[X(t-s)]ds$$

since X(t) is WSS, then m_X = E[X(t)] = E[X(t-s)].

$$m_Y = E[Y(t)] = m_X \int_{-\infty}^{\infty} h(\tau)d\tau = m_X \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau(\tau=0)}\, d\tau = m_X H(0)$$

$$R_Y(\tau) = E\big[Y(t)Y(t+\tau)\big] = E\left[\int_{-\infty}^{\infty} h(s)X(t-s)ds \int_{-\infty}^{\infty} h(r)X(t+\tau-r)dr\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)E\big[X(t-s)X(t+\tau-r)\big]dsdr = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\, R_X(\tau+s-r)dsdr$$

$$S_Y(f) = F\{R_Y(\tau)\} = \int_{-\infty}^{\infty} R_Y(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\, R_X(\tau+s-r)\, e^{-j2\pi f\tau}\, dsdrd\tau$$

Let $u = \tau + s - r$ and substitute above:

$$S_Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\, R_X(u)\, e^{-j2\pi f(u-s+r)}\, dsdrdu$$

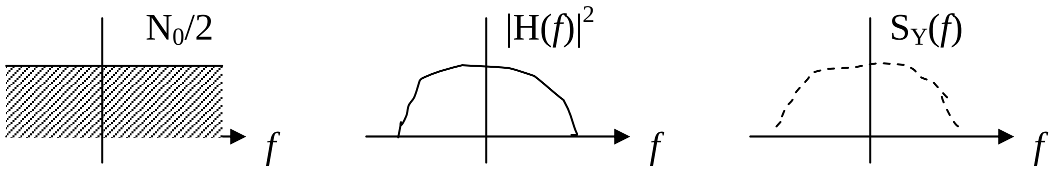
$$= \int_{-\infty}^{\infty} h(s)\, e^{j2\pi fs}\, ds \int_{-\infty}^{\infty} h(r)\, e^{-j2\pi fr}\, dr \int_{-\infty}^{\infty} R_X(u)\, e^{-j2\pi fu}\, du$$

$$= H^*(f)H(f)\, S_X(f) = |H(f)|^2 \cdot S_X(f)$$

Example: 7.9 Filtered White Noise: White noise as a signal with power spectral density (PSD) $S_X(f) = \frac{N_0}{2}$ is band-limited by a linear, time-invariant system with a frequency response: $H(f)$. What is the power spectral density of Y(t)?

$$S_Y(f) = |H(f)|^2 \frac{N_0}{2}$$

Therefore, the transfer function determines the shape of the output power spectral density.



Example: 7.11 Given: Z(t) = X(t) + Y(t) X(t) and Y(t) independent r.v.'s with power spectral density Fig 7.6(a)

Low Pass Filter output:

$$S_W(f) = \underbrace{|H_{LP}(f)|^2}_{1} S_X(f) + \underbrace{|H_{LP}(f)|^2}_{0} S_Y(f) = S_X(f)$$

$S_W(f) = S_X(f)$ but $W(t) \neq X(t)$

We can show that W(t) = X(t) in the mean square sense.

Band Pass Filter output:

$$S_W(f) = \underbrace{|H_{BP}(f)|^2}_{0} S_X(f) + \underbrace{|H_{BP}(f)|^2}_{1} S_Y(f) = S_Y(f)$$

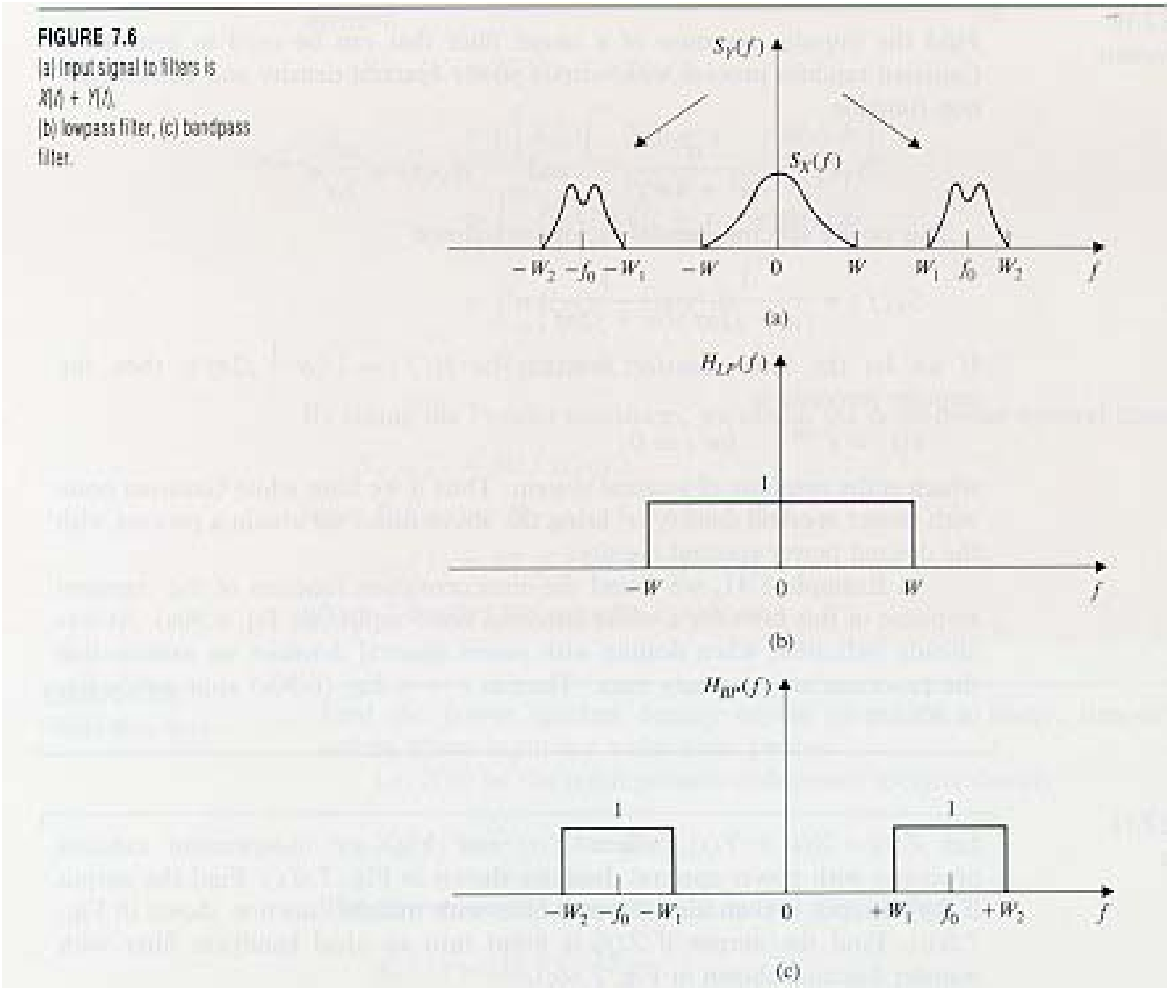


FIGURE 7.6
(a) Input signal to filters is $X(t) + Y(t)$, (b) lowpass filter, (c) bandpass filter.

Example: 7.12 Random Telegraph Signal passed through an RC low-pass filter with a transfer function:

$$H(f) = \frac{\beta}{\beta + j2\pi f} \qquad \text{where } \beta = 1/RC \text{ is the time constant}$$

$$S_Y(f) = |H(f)|^2 \, S_X(f) = \left(\frac{\beta^2}{\beta^2 + 4\pi^2 f^2}\right)\underbrace{\left(\frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}\right)}_{from\ Ex:7.1}$$

$$= \frac{4\alpha\beta^2}{\beta^2 - 4\alpha^2}\left\{\frac{1}{4\alpha^2 + 4\pi^2 f^2} - \frac{1}{\beta^2 + 4\pi^2 f^2}\right\}$$

and the inverse FT yields the autocorrelation function:

$$R_Y(\tau) = F^{-1}\{S_Y(f)\} = \frac{1}{\beta^2 - 4\alpha^2}\left\{\beta^2 \, e^{-2\alpha|\tau|} - 2\alpha\beta \, e^{-2\beta|\tau|}\right\}$$

Discrete-Time Systems

If h_n is the response of a discrete LTI system to a unit-sample input, i.e.,

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

then the response will be:

$$Y_n = X_n * h_n = \sum_{j=-\infty}^{\infty} h_j X_{n-j} = \sum_{j=-\infty}^{\infty} X_j h_{n-j}$$

The transfer function or Frequency Response of the system is given by

$$H(f) = \sum_{i=-\infty}^{\infty} h_i \, e^{-j2\pi f i}$$

If X_n is a WSS process then Y_n is also WSS with mean

$$m_Y = m_X \sum_{j=-\infty}^{\infty} h_j = m_X H(0)$$

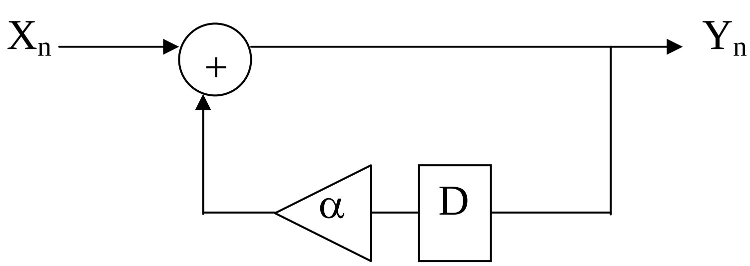
and then autocorrelation function

$$R_Y(k) = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h_j h_i \, R_X(k + j - i)$$

and Power Spectral Density of Y_n

$$S_Y(f) = |H(f)|^2 \, S_X(f)$$

Example: 7.14 First-order autoregressive process: $Y_n = \alpha Y_{n-1} + X_n$, where X_n is zero-mean white noise with average power σ_X^2 .



Unit-sample response

$$h_n = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \alpha^n & n > 0 \end{cases}$$

We need |α| < 1 for system to be stable from linear systems theory and the transfer function will be

$$H(f) = \sum_{n=0}^{\infty} \alpha^n e^{-j2\pi f n} = \frac{1}{1-\alpha e^{-j2\pi f}}$$

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{\sigma_X^2}{\left(1-\alpha e^{-j2\pi f}\right)\left(1-\alpha e^{j2\pi f}\right)}$$

$$= \frac{\sigma_X^2}{1+\alpha^2+\left(\alpha e^{-j2\pi f}+\alpha e^{j2\pi f}\right)} = \frac{\sigma_X^2}{1+\alpha^2-2\alpha\cos 2\pi f}$$

$$R_Y(k) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} h_j h_i \sigma_X^2 \delta_{k+j-i} = \sigma_X^2 \sum_{j=0}^{\infty} \alpha^j \alpha^{j+k} = \frac{\sigma_X^2 \alpha^k}{1-\alpha^2}$$

Example 7.15 Autoregressive Moving Average (ARMA) Process

$$Y_n = -\sum_{i=1}^{q} \alpha_1 Y_{n-i} + \sum_{i'=0}^{p} \beta_{i'} W_{n-i'}$$

where W_n is a WSS, white noise input process. The transfer function is

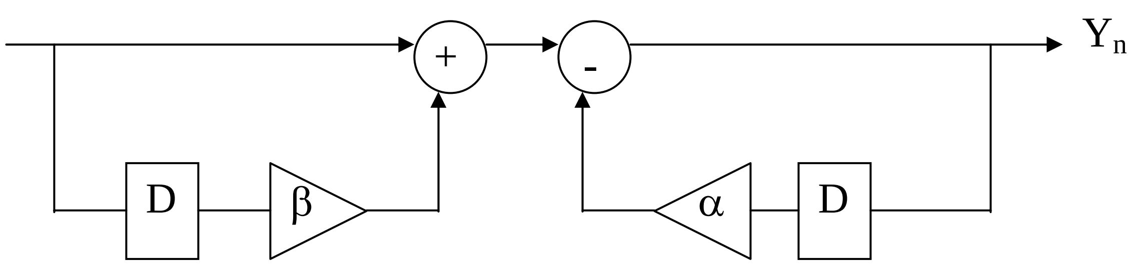
$$H(f) = \frac{\sum_{i'=0}^{p} \beta_j e^{-j2\pi f i'}}{1+\sum_{i=1}^{q} \alpha_j e^{-j2\pi f i}}$$

The power spectral density is

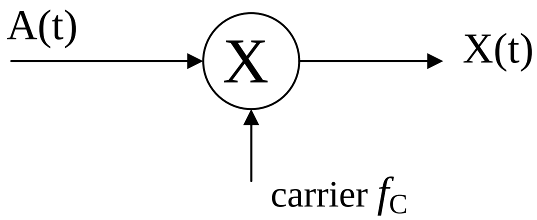
$$S_Y(f) = |H(f)|^2 \sigma_W^2$$

Special cases:

- Autoregressive (AR) Process is an ARMA process with
  $\beta_0 = 1;\ \beta_1 = \beta_2 = \cdots = \beta_p = 0$
- Moving Average (MA) process is an ARMA process with
  $\alpha_0 = 1;\ \alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$



Amplitude Modulation (AM) of Random Signals



A(t): WSS random information signal with a power spectral density $S_A(f)$ (Baseband Signal) (Fig. 7.7a)

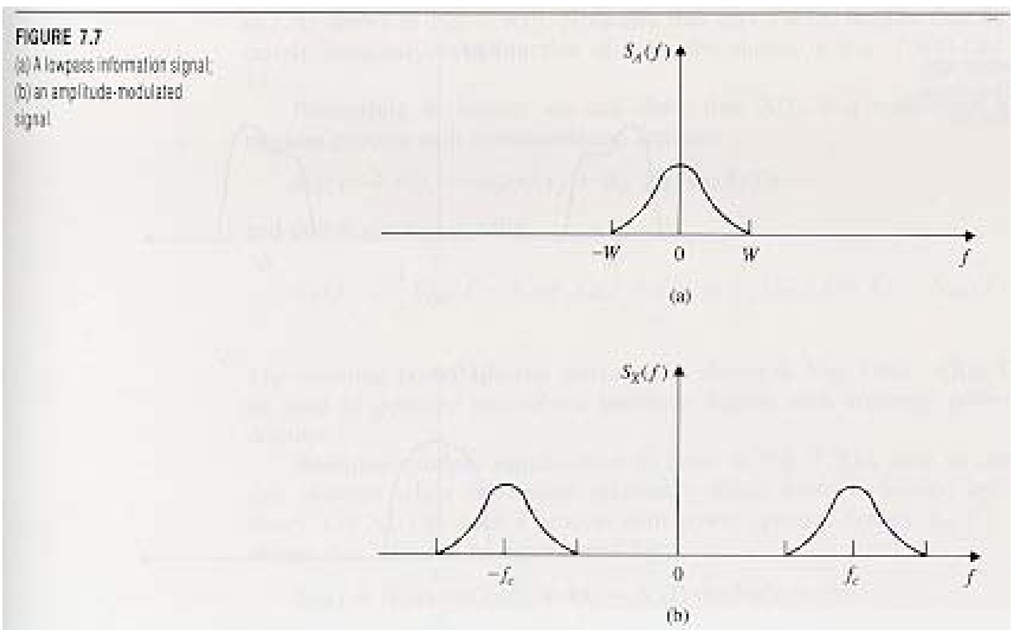


FIGURE 7.7
(a) A lowpass information signal; (b) an amplitude-modulated signal.

AM: $X(t) = A(t)\cos(2\pi f_C t + \theta)$ with A(t) & θ independent of each other And θ uniformly distributed $(0, 2\pi)$

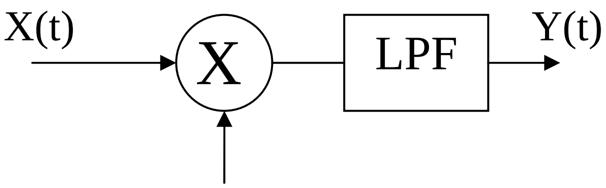
Autocorrelation:

$$R_X(\tau) = E[X(t+\tau)X(t)]$$
$$= E[A(t+\tau)\cos(2\pi f_C(t+\tau)+\theta)A(t)\cos(2\pi f_C t+\theta)]$$
$$= E[A(t+\tau)A(t)]E[\cos(2\pi f_C(t+\tau)+\theta)\cos(2\pi f_C t+\theta)]$$
$$= R_A(\tau)E\left[\frac{1}{2}\cos(2\pi f_C\tau) + \frac{1}{2}\underbrace{\cos(2\pi f_C(2t+\tau)+2\theta)}_{0}\right]$$
$$= \frac{1}{2}R_A(\tau)\cos(2\pi f_C\tau) \quad \Rightarrow X(t)\ \textit{is also WSS}$$

$$S_X(f) = F\left\{\frac{1}{2}R_A(\tau)\cos(2\pi f_C\tau)\right\} = \frac{1}{4}S_A(f+f_C) + \frac{1}{4}S_A(f-f_C)$$

(For Bandpass Signal see Figure 7.7b)

Demodulation:



$2\cos(2\pi f_C t + \theta)$

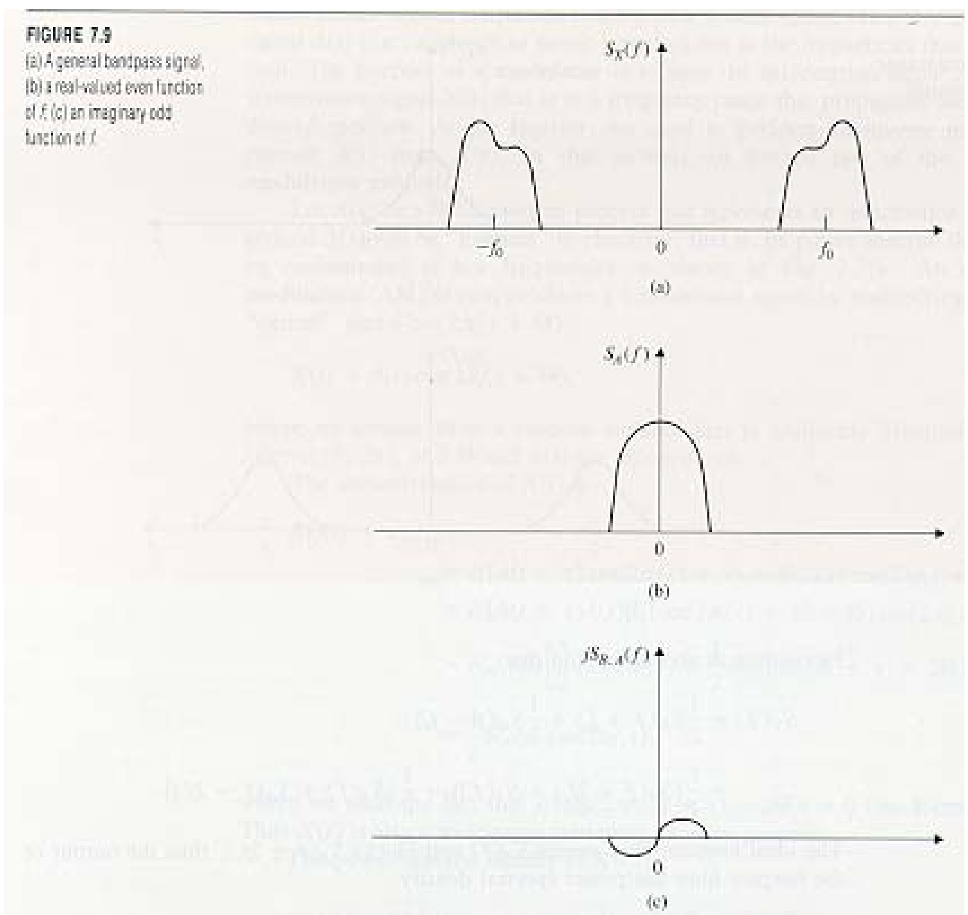
$$Y(t) = X(t)\, 2\cos(2\pi f_C t + \theta)$$

Using the above procedure we get

$$S_Y(f) = 1/2(S_X(f+f_C))+1/2\,(S_X(f-f_C))$$
$$= 1/2\{S_A(f + 2f_C) + S_A(f)\} + 1/2\{\, S_A(f) + S_A(f - 2f_C)\}$$

Let LPF be a good LPF to pass $S_A(f)$ in $-w \leq \beta < w$
But suppress $S_A(f + 2f_C)$ and $S_A(f - 2f_C)$ Then Y(t) = A(t) recovered information signal

Quadrature Amplitude Modulation (QAM)



$$X(t) = A(t)\cos(2\pi f_C t + \theta) + B(t)\sin(2\pi f_C t + \theta)$$

A(t), B(t): real-valued jointly WSS process and let

$R_A(\tau) = R_B(\tau)$

$R_{BA}(\tau) = -R_{AB}(\tau)$

$\Rightarrow \quad S_A(f) = S_B(f)$ Real-valued with even spectra (Figure 7.9b)

$\Rightarrow \quad S_{BA}(f)$ Purely imaginary (Figure 7.9c)

It is shown that X(t) is WSS with

$$R_X(\tau) = R_A(\tau)\cos(2\pi f_C \tau) + R_{BA}(\tau)\sin(2\pi f_C \tau)$$

and

$$S_X(f) = \frac{1}{2}\{S_A(f - f_C) + S_A(f + f_C)\} + \frac{1}{2j}\{S_{BA}(f - f_C) - S_{BA}(f + f_C)\}$$

WSS White Noise can be filtered by such bandpass filters

Example: 7.16 Demodulation of signal corrupted by additive noise

$$Y(t) = A(t)\cos(2\pi f_C t + \theta) + N(t)$$

where N(t): Bandlimited white noise with power spectral density

$$S_N(f) = \begin{cases} \frac{N_0}{2} & |f \pm f_C| < W \\ 0 & o.w. \end{cases}$$

We now obtain SNR of the recovered signal

$$Y(t) = [A(t) + N_C(t)]\cos(2\pi f_C t + \theta) - N_S(t)\sin(2\pi f_C t + \theta)$$

Demodulate with $2\cos(2\pi f_C t + \theta)$

$$\begin{aligned} 2Y(t)\cos(2\pi f_C t + \theta) &= \{A(t) + N_C(t)\}2\cos^2(2\pi f_C t + \theta) - N_S(t)2\cos(2\pi f_C t + \theta)\sin(2\pi f_C t + \theta) \\ &= \{A(t) + N_C(t)\}(1 + \cos(4\pi f_C t + 2\theta)) - N_S(t)\sin(4\pi f_C t + 2\theta) \end{aligned}$$

After low-pass filtering, the recovered signal is $A(t) + N_C(t)$

The signal power and noise are

$$\sigma_A^2 = \int_{-W}^{W} S_A(f)df$$

$$\sigma_{N_C}^2 = \int_{-W}^{W} S_{N_C}(f)df = \int_{-W}^{W}\left(\frac{N_0}{2} + \frac{N_0}{2}\right)df = 2WN_0$$

and the SNR is simply:

$$SNR = \frac{\sigma_A^2}{2WN_0}$$

#7.3 Find Power Spectral Density , S_Y of $R_X(\tau)\cos(2\pi f_0\tau)$

$$S_Y(f) = F[R_X(\tau)\cos 2\pi f_0\tau]$$
$$= F\left[R_X(\tau)\left[\frac{e^{j2\pi f_0\tau}+e^{-j2\pi f_0\tau}}{2}\right]\right]$$
$$= \frac{1}{2}F\left[R_X(\tau)e^{j2\pi f_0\tau}\right]+\frac{1}{2}\Im\left[R_X(\tau)e^{-j2\pi f_0\tau}\right]$$
$$= \frac{1}{2}S_X(f-f_0)+\frac{1}{2}S_X(f+f_0)$$

where

$$S_X(f) = F[R_X(\tau)].$$

#7.8 X(t) and Y(t) are independent WSS r.p. with: Z(t) = X(t)Y(t)

a) Show Z(t) is WSS

$$E[Z(t)] = E[X(t)]E[Y(t)] = m_X m_Y$$
$$R_Z(t,t+\tau) = E[X(t)X(t+\tau)Y(t)Y(t+\tau)] = E[X(t)X(t+\tau)]E[Y(t)Y(t+\tau)]$$
$$= R_X(\tau)R_Y(\tau) = R_Z(\tau)$$

Therefore, Z(t) is WSS

b) Find $R_Z(\tau)$ (shown above) and $S_Z(f)$

$$S_Z(f) = \Im[R_Z(\tau)] = \Im[R_X(\tau)R_Y(\tau)] = S_X(f)*S_Y(f)$$

#7.18 Y(t) is derivative of X(t), a bandlimited white noise process, Ex: 7.3

a) Find $S_Y(f)$ and $R_Y(\tau)$

$$S_Y(f) = |H(f)|^2 S_W(f) = |j2\pi f|^2\frac{N_0}{2} = 2\pi^2 f^2 N_0 \qquad f<W$$

$$R_Y(\tau) = \int_{-W}^{W} 2\pi^2 f^2 N_0 e^{j2\pi f\tau}df = 2\pi^2 N_0\left[e^{j2\pi f\tau}\frac{(-4\pi^2 f^2\tau^2 - 2j2\pi f\tau + 2)}{(j2\pi\tau)^3}\right]_{-W}^{W}$$

$$= 2\pi^2 N_0.[e^{j2\pi W\tau}\frac{(2-4\pi^2W^2\tau^2 - 4j\pi W\tau)}{-j8\pi^3\tau^3} - e^{-j2\pi W\tau}\frac{(2-4\pi^2W^2\tau^2 + 4j\pi W\tau)}{-j8\pi^3\tau^3}$$

$$R_Y(\tau) = \frac{4\pi^2 N_0}{8\pi^3\tau^3}\left[-(2-4\pi^2W^2\tau^2)\sin 2\pi W\tau + 4\pi W\tau\cos 2\pi W\tau\right]$$
$$= \frac{N_0}{\pi\tau^3}\left[2\pi W\tau\cos 2\pi W - (1-2\pi^2W^2\tau^2)\sin 2\pi W\tau\right]$$

b) What is the average power of the output?

$$R_Y(0) = \int_{-W}^{W} S_Y(f)df = \int_{-W}^{W} 2\pi^2 f^2 N_0 df = \frac{4\pi^2 N_0 W^3}{3}$$

#7.38 Random Telegraph Signal with transition rate α. Given $f_C = \alpha/\pi$ and $f_C = 10\alpha/\pi$. Plot the power spectral density:

$$S_X(f) = \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}$$

$$S_Y(f) = \frac{1}{2} S_X(f + f_C) + \frac{1}{2} S_X(f - f_C)$$

$$= \frac{2\alpha}{4\alpha^2 + 4\pi^2 (f + f_C)^2} + \frac{2\alpha}{4\alpha^2 + 4\pi^2 (f - f_C)^2}$$

$$= \frac{2\alpha}{4\alpha^2 + (w + 2\pi f_C)^2} + \frac{2\alpha}{4\alpha^2 + (w - 2\pi f_C)^2}$$

where w = 2π*f*